An introduction to the Universal Coefficient Theorem

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Content

Outline

[RS] p. 431

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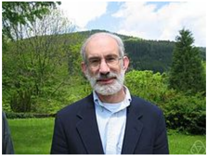
THE KÜNNETH THEOREM AND THE UNIVERSAL COEFFICIENT THEOREM FOR KASPAROV'S GENERALIZED *K*-FUNCTOR

JONATHAN ROSENBERG AND CLAUDE SCHOCHET



[RS] p. 434

difficulties, WE HENCEFORTH ASSUME THAT A IS SEPARABLE NUCLEAR AND THAT B HAS A COUNTABLE APPROXIMATE UNIT THROUGHOUT THE PAPER unless stated otherwise.

[RS] p. 439

UNIVERSAL COEFFICIENT THEOREM (UCT) 1.17. *Let $A \in \mathcal{N}$. Then there is a short exact sequence*

$$0 \to \mathrm{Ext}^1_{\mathbf{Z}}(K_*(A), K_*(B)) \xrightarrow{\delta} KK_*(A, B) \xrightarrow{\gamma} \mathrm{Hom}(K_*(A), K_*(B)) \to 0$$

which is natural in each variable. The map γ has degree 0 and the map δ has degree 1.

Brown's UCT

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(K^0(X), \mathbb{Z}) \to \mathrm{Ext}(C(X)) \to \mathrm{Hom}(K^1(X), \mathbb{Z}) \to 0$$

[RS] p. 439

Here are our principal theorems. Let $\mathcal{N}$ be the smallest full subcategory of the separable nuclear C^*-algebras which contains the separable Type I C^*-algebras and is closed under strong Morita equivalence (by [7], this is the same as stable isomorphism), inductive limits, extensions, and crossed products by **R** and by **Z**. We may also require that if J is an ideal in A and J and A are in $\mathcal{N}$ then so is A/J, and if A and A/J are in $\mathcal{N}$ then so is J. As pointed out by Skandalis [39],

[RS] p. 439

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which is natural in each variable. The map γ has degree 0 *and the map δ has degree* 1.

[RS] proof, first steps

We consider

$$\gamma(A,B) : KK_*(A,B) \to \mathrm{Hom}(K_*(A), K_*(B))$$

- If $K_*(B)$ is injective, $I \lhd A$, and two out of $\gamma(I,B)$, $\gamma(A,B)$, $\gamma(A/I,B)$ are isomorphisms, so is the last.
- If $K_*(B)$ is injective, $A = \varinjlim A_i$, and all $\gamma(A_i, B)$ are isomorphisms, so is $\gamma(A,B)$.
- If $K_*(B)$ is injective then $\gamma(C_0(X), B)$ is an isomorphism.
- If $K_*(B)$ is injective and A is type I then $\gamma(A,B)$ is an isomorphism.

[RS] proof, last steps

- If $K_*(B)$ is injective, and $A \in \mathcal{N}$, then $\gamma(A, B)$ is an isomorphism.
- For any σ-unital B there is $\varphi : B \to D$ with $K_*(D)$ injective and $\varphi_* : K_*(B) \to K_*(D)$ injective.

[RS] p. 454

It is quite possible that the UCT (1.17) holds for completely arbitrary separable C^*-algebras A. (This is assuming B has a countable approximate unit.

[RS] p. 456

An interesting open problem is to determine whether the UCT might in fact hold for *all* separable C^*-algebras. The argument of Corollary 7.5 shows that this is equivalent to the question: is every separable C^*-algebra KK-equivalent to a commutative C^*-algebra?* To obtain still another formulation, let A be any

*(Added May, 1986) Recent work of G. Skandalis now shows this is not the case, though this may be true for nuclear separable C^*-algebras.

Proposition [RS]

If $A \in \mathcal{N}$, then A is KK-equivalent to some $C_0(X)$.

Theorem [Skandalis]

The following are equivalent for a separable A (non necessarily nuclear!)

1. The UCT holds for A and any B
2. A is KK-equivalent to some $C_0(X)$
3. If $K_*(B) = 0$, then $KK(A, B) = 0$

and there is a non-nuclear A for which they are false.

Outline

Theorem [Elliott]

For A and B AT-algebras of real rank zero, we have

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \Longleftrightarrow (K_*(A), K_*(A)_+) \cong (K_*(B), K_*(B)_+)$$

The UMCT [Dadarlat-Loring]

For $A \in \mathcal{N}$ we have

$$\begin{array}{ccccc} \mathrm{Pext}(K_*(A), K_*(B)) & & & & \mathrm{Hom}(K_*(A), K_*(B)) \\ & \searrow & & \nearrow & \\ & & KK(A,B) & & \\ & \nearrow & & \searrow & \\ \mathrm{Ext}(K_*(A), K_*(B)) & & & & \mathrm{Hom}_\Lambda(\underline{K}(A), \underline{K}(B)) \end{array}$$

Theorem [Dadarlat-Loring]

For A and B AD-algebras of real rank zero, we have

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \Longleftrightarrow (\underline{K}(A), \underline{K}(A)_+) \cong (\underline{K}(B), \underline{K}(B)_+)$$

Theorem [Kirchberg-Phillips]

Suppose A and B are simple, separable, nuclear, purely infinite C^*-algebras. If A and B are KK-equivalent, then $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

Theorem [Kirchberg-Phillips]

Suppose A and B are simple, separable, nuclear, purely infinite C^*-algebras with $A, B \in \mathcal{N}$. Then

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \Longleftrightarrow K_*(A) \cong K_*(B)$$

Outline

Theorem [Tikuisis-White-Winter]

If A is separable, nuclear and satisfies the UCT, then any amenable trace on A is quasidiagonal.

Theorem [Dadarlat]

If A is separable, exact, residually finite-dimensional and satisfies the UCT, then A is AF-embeddable

New classes

A satisfies the UCT when

- $A = C^*(G)$ for certain amenable groupoids G (Tu)
- A may be locally approximated with UCT subalgebras (Dadarlat)
- $A = C^*_\pi(G)$ for a nilpotent group (Eckhart-Gillaspy)
- A has a Cartan subalgebra (Barlak-Li)

Localizations

The UCT holds for all nuclear C^*-algebras if $\mathcal{O}_2$ is unique with $K_*(\mathcal{O}_2) = 0$ among the purely infinite, nuclear C^*-algebras [Kirchberg].

Theorem [Kirchberg]

Suppose A and B are separable, nuclear, purely infinite C^*-algebras with

$$\mathrm{Prim}(A) \cong X \cong \mathrm{Prim}(B)$$

If A and B are $KK(X)$-equivalent, then $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

Theorem [Meyer-Nest,Bentmann-Köhler]

Suppose A and B are separable, nuclear, purely infinite C^*-algebras with $A, B \in \mathcal{N}$ and

$$\mathrm{Prim}(A) \cong X \cong \mathrm{Prim}(B)$$

with X a finite accordion space. Then

$$A \otimes \mathbb{K} \cong B \otimes \mathbb{K} \Longleftrightarrow K_*(X, A) \cong K_*(X, B)$$