Cartan subalgebras, automorphisms and the UCT problem

Masterclass “Applications of the UCT for C^*-algebras”
University of Copenhagen

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Theorem (B.-Szabó 2017)

The following are equivalent:

1) *every separable, nuclear C^*-algebra satisfies the UCT;*
2) *for $p = 2, 3$ and for every outer strongly approximately inner action $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$, the crossed product $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_p$ satisfies the UCT.*

This reduction of the UCT problem uses that many Kirchberg algebras (absorbing suitable UHF algebras) can be written as crossed products of $\mathcal{O}_2$ by actions of $\mathbb{Z}_2$ or $\mathbb{Z}_3$ (Izumi (2004) or B.-Szabó (2017)).

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Here $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ is said to be strongly approximately inner if $\alpha = \lim_{n\to\infty} \mathrm{Ad}(u_n)$ for some unitaries $u_n \in \mathcal{O}_2^\alpha$ (Izumi 2004).

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It is an open question whether all outer actions of $\mathbb{Z}_p$ on $\mathcal{O}_2$ are strongly approximately inner.

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(iv) there exists a faithful conditional expectation $A \to B$.

(A, B) is called a Cartan pair.

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Renault (2008) has shown that for any Cartan pair (A, B) with A separable, there exists a twisted étale, locally compact, Hausdorff groupoid (G, Σ) such that $(A, B) \cong (\mathrm{C}^*_{\mathrm{red}}(G, \Sigma), C_0(G^{(0)}))$.

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Combining this with our result and Kirchberg's reduction of the UCT problem to Kirchberg algebras, the UCT problem turns out to have an affirmative answer exactly if every Kirchberg algebra has a Cartan subalgebra.

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$$E(\mathcal{S}) = \left\{e \in \mathcal{S} \ : \ e = e^2\right\} = \left\{e \in \mathcal{S} \ : \ e = e^2 = e^*\right\}$$

denote the semi-lattice of idempotent elements and write $C^*(E(\mathcal{S}))$ for the commutative C^*-subalgebra of A generated by $E(\mathcal{S})$.

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(i) $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_n$ *satisfies the UCT;*

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$(i) \Rightarrow (ii)$: Let $\alpha : \mathbb{Z}_n \curvearrowright \mathcal{O}_2$ be outer strongly approximately inner such that $A := \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_n$ satisfies the UCT. Using the Pimsner-Voiculescu sequence for $\mathbb{Z}_n$-actions, that n is a prime power and Kirchberg-Phillips classification, one checks that $A \cong A \otimes M_{n^\infty}$.

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Using a model action result of B.-Szabó and Izumi's rigidity result for actions with the Rokhlin property, we get that

$$(A, \gamma) \cong \lim_{k \to \infty} ((C(\mathbb{Z}_n) \otimes M_n^{\otimes k-1} \otimes A, \varphi_k), \mathbb{Z}_n\text{-shift} \otimes \mathrm{id}_{M_n^{\otimes k-1} \otimes A}),$$

where $\varphi_k(f)(m) = \sum_{\ell=0}^{n-1} e_{\ell,\ell} \otimes (\mathrm{id}_{M_n^{\otimes k-1}} \otimes \beta^\ell)(f(m+\ell))$.

Idea of proof (cont.)

By a result of Izumi, the dual action $\hat{\alpha} = \gamma : \mathbb{Z}_n \curvearrowright A$ has the Rokhlin property. Combining results of Katsura and Spielberg, we find an action $\beta : \mathbb{Z}_n \curvearrowright A$ and an inverse semigroup $\tilde{\mathcal{S}} \subset A$ of partial isometries such that

- $K_*(\beta) = K_*(\gamma)$;
- $\mathrm{C}^*(\tilde{\mathcal{S}}) = A$ and $\mathrm{C}^*(E(\tilde{\mathcal{S}})) \subset A$ is a Cartan subalgebra;
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Here we use the fact that A absorbs M_{n^∞} tensorially.

Idea of proof (cont.)

Let $B \subset A$ be the abelian C^*-subalgebra corresponding to

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The following statements are equivalent:

(i) *every separable, nuclear* C^**-algebra satisfies the UCT;*

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The following statements are equivalent:

(i) *every separable, nuclear* C^**-algebra satisfies the UCT;*

(ii) *for every prime number* $p \geq 2$ *and every outer strongly approximately inner action* $\alpha : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ *there exists an inverse semigroup* $\mathcal{S} \subset \mathcal{O}_2$ *of* α*-homogeneous partial isometries such that* $\mathcal{O}_2 = \mathrm{C}^*(\mathcal{S})$ *and* $\mathrm{C}^*(E(\mathcal{S}))$ *is a Cartan subalgebra in both* $\mathcal{O}_2^\alpha$ *and* $\mathcal{O}_2$ *(with spectrum homeomorphic to the Cantor set);*

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(iii) *every outer strongly approximately inner* $\mathbb{Z}_p$*-action on* $\mathcal{O}_2$ *with* $p = 2$ *or* $p = 3$ *fixes some Cartan subalgebra* $C \subset \mathcal{O}_2$ *globally.*

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1) $KK(\alpha) = KK(\beta)$;
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In particular, all aperiodic automorphisms of $\mathcal{O}_2$ *are cocycle conjugate to each other.*

Proposition

The UCT problem has an affirmative answer if for every aperiodic automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_2)$ there exists a Cartan subalgebra $B \subset \mathcal{O}_2$ such that $\alpha(B) = B$.

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Question

Is the converse of the above statement true as well?

Proposition

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Question

Is the converse of the above statement true as well? In other words, are the following two statements equivalent:

1) every separable, nuclear C^*-algebra satisfies the UCT;
2) for every aperiodic automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_2)$ there exists a Cartan subalgebra $B \subset \mathcal{O}_2$ such that $\alpha(B) = B$?

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It suffices to show that $\alpha \otimes \mathrm{id}_{\mathcal{O}_\infty} : \mathbb{Z}_m \curvearrowright \mathcal{O}_2 \otimes \mathcal{O}_\infty \cong \mathcal{O}_2$ fixes some Cartan subalgebra (globally), as in this case $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_m$ satisfies the UCT.

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Thank you for your attention!