Irreducible representations of nilpotent groups generate classifiable C^*-algebras

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- Proof that $C^*_\pi(G)$ satisfies the UCT when G f.g. nilpotent

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- $C^*_\pi(G) \cong pC^*(G/Z(G), \omega)$
- Structure of $C^*_\pi(H)$ when H is virtually nilpotent

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$\square$

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Let $G = UT(4,\mathbb{Z})$, $\theta \in (0,1)\backslash\mathbb{Q}$, Θ the trace on $UT(4,\mathbb{Z})$

$$\Theta\begin{pmatrix}1 & a & b & c\\ 0 & 1 & d & e\\ 0 & 0 & 1 & f\\ 0 & 0 & 0 & 1\end{pmatrix} = \begin{cases} e^{2\pi i c\theta}, & a=b=d=e=f=0\\ 0, & \text{else.}\end{cases}$$

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Moreover, [EKM16] $C^*_{\pi_\Theta}(G) \cong C^*(G/Z(G), \omega_\theta) \cong (A_\theta \otimes A_\theta) \rtimes_\beta \mathbb{Z}$.

The twist is given by $\omega_\theta(\gamma, \eta) = \Theta(GH(HG)^{-1})$, where G is a representative of γ with $c = 0$.

A quick advertisement of classification

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Since the algebras $C^*_{\pi_\Theta}(G)$ are classifiable, these isomorphisms give us new perspectives on $C^*_{\pi_\Theta}(G)$.

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Theorem (Eckhardt-G)

If τ is a trace on a f.g. nilpotent group G such that $\tau(x) \neq 0 \Rightarrow x \in G_f$ (x has a finite conjugacy class), then

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Twisted group C^*-algebras

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For G f.g. nilpotent and π a faithful irrep of G, $\exists N \leq Z(G)$ s.t.

$$C^*_\pi(G) \cong pC^*(G/N, \sigma)$$

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    - $\pi_\tau \prec \pi_\omega$;
    - Conditional expectations $E_i : C^*_i(G) \to C^*_i(G_f)$ for $i = \pi_\tau, \pi_\omega$.

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- $C^*_{\pi_\omega}(G) \cong C^*(G/N, \sigma)$ where

$$\sigma(sN, tN) = \omega(c(sN)c(tN)c(stN)^{-1})$$

for a choice $c : G/N \to G$ of coset representatives [EKM16]

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- Since $N \cong \mathbb{Z}^d$ and $\omega : N \to \mathbb{T}$,

$$\omega \in \widehat{\mathbb{Z}^d} \cong \mathbb{T}^d$$

  which is path connected. $\square$

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- $C^*_\pi(G)$ is a direct summand of $\left(C^*_{\pi_\tau}(N) \rtimes_{\alpha,\omega} H/N\right) \otimes M_{G/H}$
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- $C^*_{\pi_\tau}(N)$ is a direct sum of simple $\mathcal{Z}$-stable C^*-algebras.
- If H/N is simple, the twisted action (α, ω) must be either strongly outer, or inner.

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