STRATIFICATION OF MODULAR REPRESENTATIONS OF FINITE GROUPS

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The set-up

- G a finite group
- k a field of characteristic $p > 0$ dividing the order of G
- $\mathsf{Mod}\, kG$ the category of modules over the group algebra kG
- $\mathsf{StMod}\, kG$ the stable category (a compactly generated triangulated category)
- $M \otimes_k N$ the tensor product of kG-modules (with diagonal G-action)
- $\mathrm{Hom}_k(M, N)$ the function object of kG-modules (with diagonal G-action)
- $H^*(G, k) = \mathrm{Ext}^*_{kG}(k, k)$ the group cohomology algebra (graded commutative and noetherian)
- $\mathrm{Proj}\, H^*(G, k)$ the set of homogeneous prime ideals not containing $H^{\geq 1}(G, k)$

1. Local cohomology functors

Let M, N be kG-modules. Then

$$\widehat{\mathrm{Ext}}^*_{kG}(M, N) \cong \bigoplus_{i \in \mathbb{Z}} \underline{\mathrm{Hom}}_{kG}(M, \Omega^{-i} N)$$

is a $H^*(G, k)$-module via

$$H^*(G, k) = \mathrm{Ext}^*_{kG}(k, k) \xrightarrow{-\otimes_k M} \mathrm{Ext}^*_{kG}(M, M) \longrightarrow \widehat{\mathrm{Ext}}^*_{kG}(M, M).$$

Fact. *Let $\mathfrak{p} \in \mathrm{Proj}\, H^*(G, k)$. Then there is an exact functor*

$$\mathsf{StMod}\, kG \longrightarrow \mathsf{StMod}\, kG, \quad N \mapsto N_{\mathfrak{p}}$$

and a natural morphism $N \to N_{\mathfrak{p}}$ inducing for all finite dimensional M an isomorphism

$$\widehat{\mathrm{Ext}}^*_{kG}(M, N)_{\mathfrak{p}} \xrightarrow{\sim} \widehat{\mathrm{Ext}}^*_{kG}(M, N_{\mathfrak{p}}).$$

An object M is $\mathfrak{p}$*-torsion* if $M_{\mathfrak{q}} = 0$ for all $\mathfrak{p} \not\subseteq \mathfrak{q}$.

Fact. *Let $\mathfrak{p} \in \mathrm{Proj}\, H^*(G, k)$. Then the full subcategory of $\mathfrak{p}$-torsion objects admits a right adjoint $M \mapsto \varGamma_{\mathcal{V}(\mathfrak{p})} M$.*

The *local cohomology functor* $\mathsf{StMod}\, kG \to \mathsf{StMod}\, kG$ is given by

$$\varGamma_{\mathfrak{p}} M := \varGamma_{\mathcal{V}(\mathfrak{p})}(M_{\mathfrak{p}}) \cong (\varGamma_{\mathcal{V}(\mathfrak{p})} M)_{\mathfrak{p}}.$$

2. Cohomological support and cosupport

For a kG-module M define

$$\operatorname{supp}_G(M) := \{\mathfrak{p} \in \operatorname{Proj} H^*(G,k) \mid \varGamma_{\mathfrak{p}} k \otimes_k M \text{ is not projective}\}$$

and

$$\operatorname{cosupp}_G(M) := \{\mathfrak{p} \in \operatorname{Proj} H^*(G,k) \mid \operatorname{Hom}_k(\varGamma_{\mathfrak{p}} k, M) \text{ is not projective}\}.$$

3. Local-global principle

Fact. *For a kG-module M, the localising subcategory of $\mathsf{StMod}\, kG$ generated by M is the same as that generated by*

$$\{\varGamma_{\mathfrak{p}} M \mid \mathfrak{p} \in \operatorname{Proj} H^*(G,k)\}.$$

4. π-points

A *π-point* of G, defined for a field extension K/k, is a morphism of K-algebras

$$\alpha\colon K[t]/(t^p) \longrightarrow KG$$

that factors through the group algebra KE of an elementary abelian p-subgroup $E \le G$, and such that KG is flat when viewed as a module over $K[t]/(t^p)$ via α.

A π-point $\alpha\colon K[t]/(t^p) \to KG$ induces a functor

$$\alpha^*\colon\ \mathsf{Mod}\, KG \longrightarrow \mathsf{Mod}\, K[t]/(t^p).$$

For π-points α and β set $\alpha \sim \beta$ if for all $M \in \mathsf{Mod}\, kG$

$$\alpha^*(K \otimes_k M) \quad \text{projective} \quad \iff \quad \beta^*(K \otimes_k M) \quad \text{projective.}$$

Let $\Pi(G)$ denote the set of equivalence classes of π-points of G.

5. π-support and π-cosupport

For a kG-module M define

$$\pi\text{-}\operatorname{supp}_G(M) := \{[\alpha] \in \Pi(G) \mid \alpha^*(K \otimes_k M) \text{ is not projective}\}$$

and

$$\pi\text{-}\operatorname{cosupp}_G(M) := \{[\alpha] \in \Pi(G) \mid \alpha^*(\operatorname{Hom}_k(K, M)) \text{ is not projective}\}.$$

6. Tensor product formula and cosupport formula

Fact. *For kG-modules M, N we have*

$$\pi\text{-}\operatorname{supp}_G(M \otimes_k N) = \pi\text{-}\operatorname{supp}_G(M) \cap \pi\text{-}\operatorname{supp}_G(N)$$

and

$$\pi\text{-}\operatorname{cosupp}_G(\operatorname{Hom}_k(M,N)) = \pi\text{-}\operatorname{supp}_G(M) \cap \pi\text{-}\operatorname{cosupp}_G(N).$$

7. π-support and π-cosupport detect projectivity

Fact. *A kG-module M is projective if and only if for every elementary abelian p-subgroup $E \le G$ the kE-module $M{\downarrow_E}$ is projective.*

Fact. *For any kG-module M we have*

$$M \text{ is projective} \quad \iff \quad \pi\text{-}\operatorname{supp}_G(M) = \varnothing \quad \iff \quad \pi\text{-}\operatorname{cosupp}_G(M) = \varnothing.$$

8. Quillen stratification

Any subgroup $H \leq G$ induces a k-algebra homomorphism

$$\operatorname{res}_{G,H} \colon H^*(G,k) \longrightarrow H^*(H,k)$$

and therefore a map

$$\operatorname{res}^*_{G,H} \colon \operatorname{Proj} H^*(H,k) \longrightarrow \operatorname{Proj} H^*(G,k).$$

Fact. *We have*

$$\operatorname{Proj} H^*(G,k) = \bigcup_{E \leq G} \operatorname{Im} \operatorname{res}^*_{G,E}$$

where E runs through all elementary abelian p-subgroups.

9. The space of π-points

A π-point $\alpha\colon K[t]/(t^p) \to KG$ induces a k-algebra homomorphism

$$H^*(\alpha)\colon H^*(G,k) = \operatorname{Ext}^*_{kG}(k,k) \xrightarrow{K\otimes_k -} \operatorname{Ext}^*_{KG}(K,K) \xrightarrow{\alpha^*} \operatorname{Ext}^*_{K[t]/(t^p)}(K,K).$$

Fact. *The assignment $\alpha \mapsto \sqrt{\operatorname{Ker} H^*(\alpha)}$ induces a bijection $\Pi(G) \xrightarrow{\sim} \operatorname{Proj} H^*(G,k)$, which we view as identification.*

10. π-(co)support equals cohomological (co)support

Fact. *For any kG-module M we have*

$$\operatorname{cosupp}_G(M) = \pi\text{-}\operatorname{cosupp}_G(M) \quad \textit{and} \quad \operatorname{supp}_G(M) = \pi\text{-}\operatorname{supp}_G(M).$$

11. Minimal localising subcategories

Fact. *For $\mathfrak{p} \in \operatorname{Proj} H^*(G,k)$ the localising subcategory*

$$\{M \in \mathsf{StMod}\, kG \mid \operatorname{supp}_G(M) \subseteq \{\mathfrak{p}\}\}$$

admits no proper non-zero tensor ideal localising subcategory.

12. Stratification

Fact. *The assignment*

$$\mathsf{C} \longmapsto \bigcup_{M \in \mathsf{C}} \operatorname{supp}_G(M)$$

induces a one to one correspondence between the tensor ideal localising subcategories of $\mathsf{StMod}\, kG$ and the subsets of $\operatorname{Proj} H^*(G,k)$.

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