Exercises for Wednesday

As usual, throughout G will be finite group, k a field of positive characteristic p dividing $|G|$, and $R := H^*(G,k)$, the cohomology ring of G.

(1) Given $r \in R^{\geqslant 1}$, prove that for any kG-module M, the kG-module $M /\!\!/ r$ is $\mathcal{V}(r)$-torsion and that

$$\operatorname{supp}_G(M /\!\!/ r) = \operatorname{supp}_G M \cap \mathcal{V}(r)\,.$$

Extend this to ideals in R.

(2) Use the preceding exercise to prove that any closed subset of $\operatorname{Proj} R$ is the support of some finite dimensional kG-module.

(3) For any $\mathfrak{p}$ in $\operatorname{Proj} R$ and kG-modules M, N, the natural map between cohomology and Tate cohomology induces an isomorphism of $R_{\mathfrak{p}}$-modules

$$\operatorname{Ext}^*_{kG}(M,N)_{\mathfrak{p}} \longrightarrow \widehat{\operatorname{Ext}}^*_{kG}(M,N)_{\mathfrak{p}}$$

This is another reason why it does matter that one works with $H^*(G,k)$ action on $\mathsf{StMod}\,kG$ rather than that of the Tate cohomology ring.

(4) Let $E = (\mathbb{Z}/3)^2$, the elementary abelian 3-group of rank 2, and k a field of characteristic three. Its cohomology is of the form $\wedge_k(x_1,x_2) \otimes_k k[y_1,y_2]$, with $|x_i| = 1$ and $|y_i| = 2$.

Describe $\Omega^{-1}k$ and $\Omega^{-2}k$, and the classes x_i and y_i. Use this and the recipe outlined in today's first lecture to describe $\mathit{\Gamma}_{(0)}k$.

(5) Let k be a field of characteristic two, and

$$R := k[x,y,z]/(x^2+xy+y^2, x^2y+xy^2),$$

where $|x| = 1 = |y|$ and $|z| = 4$. This is the cohomology of Q_8; check this. Verify that $\operatorname{Proj} R$ consists of a single point, namely, (x,y).

(6) Let k be a field of odd characteristic and R the ring $k[x,y,z]/(x^2+y^2+z^2)$. Prove that this is a domain and find a generic point for (0).