M

StMod

Talk 4 (Greenlees)

Duality & singularity for C^*BG

G finite group, k field of char p.

Pre-point: C^*BG is Gorenstein & it is useful

1. C^*BG

$$\underset{\overset{\parallel}{R}}{C^*BG} \longrightarrow k$$

Work in context where R is a comm. ring, so that $\mathcal{D}(R)$ is $\otimes$-Δed.

$\boxed{\text{"Honesty box": } C^*BG = \mathrm{map}(BG, Hk) \text{ (comm. ring spectrum}}$

2. Morita theory

$$\mathcal{E} := R^! := \mathrm{Hom}_{C^*BG}(k,k) \simeq C_*(\Omega(BG_p^\wedge)) \underset{\text{when } G \text{ a } p\text{-gp.}}{\simeq} kG$$

$$\begin{tikzcd} R\text{-modules} \arrow[r, bend right, "\mathrm{Hom}_R(k,-)"'] \arrow[r, phantom, "\perp"] & \operatorname{mod} \mathcal{E} \arrow[l, bend right, "-\otimes_{\mathcal{E}} k"'] \end{tikzcd}$$

Lemma: $\mathrm{Hom}_R(k,M) \otimes_{\mathcal{E}} k \xrightarrow{\varepsilon} M$ is k-cellularization

Proof: $\mathrm{Hom}_R(k,M) \vdash \mathcal{E}$ so $\mathrm{Hom}_R(k,M) \otimes_{\mathcal{E}} k \vdash \mathcal{E} \otimes_{\mathcal{E}} k = k$

"built from"

$\ulcorner M \overset{+}{\vdash} N$ = "N built from M using coprods & Δs"

$N \in \mathrm{Loc}(M)$

$M \models N$ = "N is finitely built from M using Δs & retracts"

$N \in \mathrm{Thick}(M) \lrcorner$

If $k \vdash M$ then $\mathcal{E}$ is an equivalence

e.g. $M = k$, $\mathcal{E} \otimes_{\mathcal{E}} k \xrightarrow{\simeq} k$ + $\{M \mid \mathcal{E}_M$ is an equiv$\}$ is closed under Δs & if k is compact, it's closed under $\oplus$.

(If G is a p-gp, then $R \models k$) ✓

<u>Pf</u>: $k \models kG$

$$\underbrace{\mathrm{Hom}_{kG}(k,k)}_{= R} \models \mathrm{Hom}_{kG}(kG, k) \simeq k \quad ✓$$

$\square$

<u>3. Gorenstein & Gorenstein duality</u>

<u>Def^n</u>: $R \downarrow k$ is <u>Gorenstein</u> if $\mathrm{Hom}_R(k,R) \simeq \Sigma^a k$

<u>Theorem</u>: (Dwyer, Greenlees, Iyengar)

$C^*BG \to k$ is Gorenstein.

<u>**Proof for P a p-gp**</u>

Gorenstein is Morita invariant:

$$\mathrm{Hom}_R(k, R^{\vee\vee}) \simeq \mathrm{Hom}_k(k \otimes_R R^\vee, k)$$

$\ulcorner R = C^*BG,\ M^\vee = \mathrm{Hom}_k(M, k),\ R^\vee = C_*BG \lrcorner$

$$\simeq \mathrm{Hom}_k(k \otimes_R k \otimes_{kG} k, k)$$

$$\simeq \mathrm{Hom}_k(kG^\vee \otimes_{kG} k, k)$$

$$\simeq \mathrm{Hom}_{kG}(k, kG^{\vee\vee})$$

$$\simeq k \qquad \square$$

<u>Def^n</u>: $R \to k$ has <u>Gorenstein duality</u> if

$$\mathrm{cell}_k R \simeq \Sigma^a R^\vee$$

<u>Remark</u>: $\mathfrak{m} \triangleleft H^*BG$ maximal, then

$(x_1, \ldots, x_n)$

$\cancel{\mathrm{cell}_k R \simeq (R \to R[\tfrac{1}{x_1}]) \otimes_R}$

$$\mathrm{cell}_k R \simeq \mathrm{fibre}(R \to R[\tfrac{1}{x_1}]) \otimes_R \cdots \otimes_R \mathrm{fibre}(R \to R[\tfrac{1}{x_n}])$$

e.g. $\mathrm{cell}_{\mathbb{Z}/p} \mathbb{Z} = \mathrm{fibre}(\mathbb{Z} \to \mathbb{Z}[\tfrac{1}{p}])$

$$\simeq \Sigma^{-1} \mathbb{Z}/p^\infty$$

$$\simeq \xrightarrow[s]{\mathrm{hocolim}} \Sigma^{-1} \mathbb{Z}/p^s$$

<u>Cor</u>: $R = C^*BG$ has Gor. duality

<u>Pf</u>:
$$\begin{array}{c} \mathrm{Hom}_R(k,R) \simeq k \simeq \mathrm{Hom}_R(k,R^\vee) \\ \wr\! \mid \\ \mathrm{Hom}_R(k, \mathrm{cell}_k R) \end{array}$$

Apply lemma: $\otimes_{\mathcal{E}} k$ gives $\mathrm{cell}_k R \simeq R^\vee = C_*BG$.

(lemma applies since $\exists! \, \mathcal{E}$-action on k, as there is a unique kP-mod structure on k). $\square$

4. Normalisation & $D^b(C^*BG)$

Pick a faithful rep^n $G \to U(n)$ & so

$$\begin{tikzcd} C^*BG \arrow[d, equal] & C^*BU(n) \arrow[l, "q"'] \arrow[d, equal] \\ R & S \arrow[l, "q"] \end{tikzcd}$$

<u>Def^n</u>: <u>normalisation</u> of R is a map $q : S \to R$ s.t. R is small $/S$, k is small$/S$.

<u>Example</u>: As above

<u>Def^n</u>: Given a normalisation, define $D^{q\text{-}b}(R) = \{ M \mid M \text{ an } R\text{-mod s.t } q^*M \text{ is small}/S \}$

<u>Lemma</u>: If H^*S is poly. then an S-module M is small $\iff H^*M$ is f.g. / H^*S.

$$\begin{array}{ccccc} S & \xrightarrow{\varepsilon} & R & \xrightarrow{p} & Q = R \otimes_S k \\ \| & & \| & & \cong \\ C^*BU(n) & & C^*BG & & C^*(U(n)/G) \end{array}$$

Eilenberg-Moore SS.

$$\begin{array}{ccccc} \mathrm{Hom}_S(k,k) & \xleftarrow{j} & \mathrm{Hom}_R(k,k) & \xleftarrow{i} & \mathrm{Hom}_Q(k,k) \\ \cong & & \| & & \cong \\ C_*\Omega BU(n) & & C_*\Omega BG_{\hat{p}} & & C_*(\Omega(U(n)/G)) \\ \| & & & & \\ C_*BU(n) & & & & \end{array}$$

Note: all Gorenstein, and q & i are relatively Gorenstein.

<u>Theorem</u>: (G-S): $D^{q-b}(R) \simeq D^{i-b}(\mathcal{E})$

$$\text{Hence } D^q_{sg}(R) := \frac{D^{q-b}(R)}{\mathrm{Perf}} \simeq \frac{D^{i-b}(\mathcal{E})}{\text{fin dim}^l} =: \underset{\substack{\| \\ D^b(\mathrm{Proj}\,\mathcal{E})}}{D^{\varepsilon}_{cosg}}(\mathcal{E})$$

$$D^q_{cosg}(R) \simeq D_{sg}(\mathcal{E})$$

<u>Example</u>: G a p-gp

$$\underset{\substack{\| \\ D^b(\mathrm{Proj}(C^*BG))}}{D_{cosg}(C^*BG)} \simeq D_{sg}(kG) = \mathrm{stmod}\, kG$$

<u>Theorem:</u> If G is a p-group, then a module M over $C^*BG = R$ is small $\iff$ H^*M is f.g. / H^*BG.

<u>Proof:</u> $S \xrightarrow{q} R$. If $S \models q^*M$ want $R \models M$

$q^*M \leftarrow\!\shortmid M$

$FY := \mathrm{Hom}_S(k, Y)$

$\mathcal{F} = \mathrm{Hom}_S(k,k)$

$EX = \mathrm{Hom}_R(k, X)$

$\mathcal{E} = \mathrm{Hom}_R(k,k)$

$FS \models Fq^*M \cong j_* EM$ exercise

$= \mathcal{F} \otimes_{\mathcal{E}} EM$

Apply j^*:

$j^*FS \models j^*\mathcal{F} \otimes_{\mathcal{E}} EM$

$\models \mathcal{E} \otimes_{\mathcal{E}} EM = EM$

Thomason-localisation.

$$\begin{aligned} &\mathrm{Hom}_S(k, S) \\ &\quad \shortparallel \\ &\mathrm{Hom}_R(k, \mathrm{Hom}_S(R,S)) \\ &\quad \shortparallel \leftarrow q \text{ rel Gor.} \\ &ER \end{aligned}$$

Apply $\mathrm{Hom}_{\mathcal{E}}(k, -)$ to deduce $R \models M$ $\square$.