StMod – Lecture 6 (Henning)

Auslander-Reiten / Tate / Serre duality

Auslander-Reiten duality

k a field, A a k-algebra.

Let $D = \hom_k(-, k)$, $\rightsquigarrow$ duality

$$\mathrm{Mod}\, A \underset{D}{\overset{D}{\rightleftarrows}} \mathrm{Mod}(A^{op})$$

[i.e. right modules

Thm (Auslander-Reiten, 1976)

For $X \in \mathrm{mod}\, A$ finitely presented,

$$D\underline{\mathrm{Hom}}_A(X, -) \cong \mathrm{Ext}^1_A(-, D\mathrm{Tr}\, X)$$

where $\mathrm{Tr}\, X$ = transpose of $X \in \mathrm{mod}\, A$:

[$\mathrm{mod}\, A$ = cat. of fin. pres. A-mods]

$$P_1 \to P_0 \to X \to 0 \qquad P_i \text{ f.g. proj}$$

$$\rightsquigarrow P_0^* \to P_1^* \to \mathrm{Tr}\, X \to 0, \qquad P_i^* := \mathrm{Hom}_A(P_i, A).$$

~~by definit~~ defining $\mathrm{Tr}\, X$

A-R formula is a consequence of the <u>defect formula</u>:

For an exact sequence in $\mathrm{Mod}\, A$:

$$\xi : \; 0 \to L \to M \to N \to 0$$

define

$$0 \to \mathrm{Hom}_A(N, -) \to \mathrm{Hom}_A(M, -) \to \mathrm{Hom}_A(L, -) \to \xi_* \to 0$$

(<u>covariant defect</u>) and

$$0 \to \operatorname{Hom}_A(-,L) \to \operatorname{Hom}_A(-,M) \to \operatorname{Hom}_A(-,N) \to \xi^* \to 0$$
(contravariant defect)

<u>Thm</u> (Auslander):
There is an isom
$$D\, \xi^*(X) \cong \xi_*(D\operatorname{Tr} X)$$

<u>Proof of A-R formula</u>:

Choose $\xi = 0 \to L \to M \to N \to 0$ with M projective. Then
$\xi^* \cong \underline{\operatorname{Hom}}_A(-,N)$, and
$\xi_* \cong \operatorname{Ext}^1_A(N,-)$. Apply Auslander's thm. $\square$

<u>Tate duality</u>:
A a fin. dim. k-alg. Suppose that A is <u>self-injective</u>, i.e. A is an injective A-mod.

<u>Example</u>: kG, G finite.

Nakayama functor $\nu : \operatorname{Mod} A \to \operatorname{Mod} A$
$$M \longmapsto D(A) \otimes_A M$$

<u>Lemma</u>: For $M \in \operatorname{mod} A$, $M^* := \operatorname{Hom}_A(M,A)$
$$\nu(M) \underset{(1)}{\cong} D(M^*) \underset{(2)}{\cong} \Omega^{-2}(D\operatorname{Tr} M)$$

<u>Proof</u>: (1) clear since both functors

are right-exact and agree on A_A.

(2) $P_1 \to P_0 \to M \to 0$

$\rightsquigarrow \quad 0 \to M^* \hookrightarrow P_0^* \to P_1^* \to \mathrm{Tr}\, M \to 0$

Apply D:

$0 \to D\,\mathrm{Tr}\, M \to DP_1^* \to DP_0^* \to DM^* \to 0$.

<u>Thm</u> (Tate duality)

For $M \in \mathrm{mod}\, A$ we have

$$D\,\underline{\mathrm{Hom}}_A(M, -) \cong \underline{\mathrm{Hom}}(-, \Omega \nu M)$$

<u>Proof</u>: $D\,\underline{\mathrm{Hom}}_A(M,-) \underset{AR}{\cong} \mathrm{Ext}_A^1(-, D\,\mathrm{Tr}\, M)$

$$= \underline{\mathrm{Hom}}_A(-, \Omega^{-1}(D\,\mathrm{Tr}\, M))$$

$$\underset{\text{Lemma}}{\cong} \underline{\mathrm{Hom}}_A(-, \Omega \nu M)$$

□

Note that when $A = kG$, A is a symmetric algebra in the sense that $A \cong DA$ as A-A-bimods, so ν is the identity.

<u>Serre duality</u>

<u>Def^n</u> (Bondal-Kapranov, 1990)

$\mathcal{T}$ a k-linear Hom-finite triangulated cat., where Hom-finite means $\mathrm{Hom}_{\mathcal{T}}(X, Y)$ has

finite length $/k$ for all X, Y. A **Serre functor** is an equivalence
$F: \mathcal{T} \xrightarrow{\sim} \mathcal{T}$ with natural isoms
$$D\,\mathrm{Hom}_{\mathcal{T}}(X,Y) \cong \mathrm{Hom}_{\mathcal{T}}(Y, FX) \quad \text{for all } X, Y \in \mathcal{T}$$

⌜i.e. $D\,\mathrm{Hom}_{\mathcal{T}}(X,-)$ is corepresented by FX⌟

Cor: ~~StMod~~ stmod kG has Serre duality with Serre functor $M \mapsto \Omega M$ (using Tate duality).

Aim: Prove a $\mathfrak{p}$-local version of Serre duality for stmod kG and $\mathfrak{p} \in \mathrm{Proj}\, H^*(G,k)$. This involves $(\Gamma_{\mathfrak{p}}\, \mathrm{StMod}\, kG)^c$ (i.e. $\exists$ a Serre functor for stmod kG)

⌜once you know there exists a Serre functor, it is uniquely determined by its defining property⌟

$\mathcal{C}$ = an essentially small Δed cat. with a central R-action.
(e.g. $\mathcal{C}$ = stmod kG, $R = H^*(G,k)$)
Define category $\mathcal{C}_{\mathfrak{p}}$ for $\mathfrak{p} \in \mathrm{Spec}\, R$:
obs = Ob $\mathcal{C}$
$\mathrm{Hom}^*_{\mathcal{C}_{\mathfrak{p}}}(X,Y) := \mathrm{Hom}^*_{\mathcal{C}}(X,Y)_{\mathfrak{p}}$.

Canonical functor $\mathcal{C} \to \mathcal{C}_{\mathfrak{p}}$ Δ functor, and $\mathcal{C}_{\mathfrak{p}}$ carries a canonical Δed struct.

$\gamma_{\mathfrak{p}} \mathcal{C} :=$ full subcat. of $\mathfrak{p}$-torsion objects.

$\ulcorner$ X $\mathfrak{p}$-torsion if $\mathrm{Hom}^*(-, X)$ is $\mathfrak{p}$-torsion over R $\lrcorner$

<u>Lemma</u>: Let $\mathcal{T}$ be a compactly generated R-linear Δed cat.

(1) $(\mathcal{T}^c)_{\mathfrak{p}} \xrightarrow{\sim} (\mathcal{T}_{\mathfrak{p}})^c$ up to direct summands (i.e. after idempotent completion: $(\mathcal{T}^c)_{\mathfrak{p}}$ is not necessarily idempotent complete)

(2) $\gamma_{\mathfrak{p}}(\mathcal{T}^c) \xrightarrow{\sim} (\Gamma_p \mathcal{T})^c$ up to direct summands.