St Mod - Lecture 5 (Srikanth)

Distinctive features of $H^*(G;k)$:

- it's graded commutative (f.g.) k-algebra
- want Proj, not Spec, because
  $H^{\geq 1}(G;k) \notin \operatorname{supp}(\operatorname{StMod} kG)$.

so ~~use~~ we need to adapt techniques from generic points lecture.

$\longleftrightarrow$

Let A be ~~a~~ such a k-alg, $\mathfrak{p} \in \operatorname{Proj} A$. Choose $\underline{a} = a_0, \dots, a_d$ in A s.t.

⓪ $|a_i|$ same for all i

① $\underline{a}$ alg. ind. $/k$ mod $\mathfrak{p}$

② $k[\underline{a}] \hookrightarrow A/\mathfrak{p}$ finite extension.

$K = k(t_1, \dots, t_d)$ and set $B = K \otimes_k A$,
$b_i = a_i - a_0 t_i$, $1 \leq i \leq d$, $m := \sqrt{\mathfrak{p}B + (\underline{b})B}$

__Theorem:__

① $m \in \operatorname{Proj} B$, a closed point, and $m \cap A = \mathfrak{p}$.

② $\forall\, M \in \operatorname{grMod} A$, the seq. $\underline{b}$ is weakly $\mathcal{U}_{a_0}^{-1}(B \otimes_A M)$-regular, where $\mathcal{U}_{a_0} = \{a_0^n \mid n \geq 0\}$.

③ For M $\mathfrak{p}$-torsion, the map $M \to B \otimes_A M$ induces $M_{\mathfrak{p}} \xrightarrow{\sim} \dfrac{(B \otimes_A M)}{\underline{b}(B \otimes_A M)}$.

__Comments:__ $A \longrightarrow A[1/a_0] = \mathcal{U}_{a_0}^{-1} A$

$$\begin{tikzcd} \rightsquigarrow \mathfrak{p}_0 \quad \operatorname{Spec}(\mathcal{U}_{a_0}^{-1} A) \arrow[r, hook] & \operatorname{Proj} A \\ \operatorname{Spec}(\mathcal{U}_{a_0}^{-1} B) \arrow[u] \arrow[r, hook] & \operatorname{Proj} B \arrow[u] \end{tikzcd}$$

Generic points in affine case $\rightsquigarrow$ generic points for Proj, but need to do work to check the properties we care about are preserved.

(2) Need to localise to get (weakly) regular:
(N.B. radical $\sqrt{-}$ is also necessary)

<u>Example</u>: $k[x,y]$, $|x| = |y| = 1$.
$\mathfrak{p} = (0)$, $\underline{a} = x, y$

$$k[x,y] \hookrightarrow k(t)[x,y]$$
$$\mathfrak{m} = (y - xt)$$

$$k = A/(x,y) \qquad B \otimes_A k = k(t)$$
$$y - xt = 0 \text{ on } k(t)$$

$$\longleftarrow\!\!\longrightarrow$$

(B) G a finite group, k field, $R := H^*(G,k)$ acting on $\operatorname{StMod} kG$.

<u>Def^n</u>: $r \in R$ and $M \in \operatorname{StMod} kG$ gives exact triangle ~~morphism~~ $M \xrightarrow{r} \Omega^{-|r|} M \longrightarrow M/\!\!/r \longrightarrow$

<u>Koszul</u> object.

$\Omega^{|r|}(M/\!\!/r)$ is the ~~[illegible]~~ <u>Carlson module</u>
$L_r(M)$

For any $C \in \mathrm{StMod}\, kG$, one has

$$\widehat{\mathrm{Ext}}^*_G(C,M) \xrightarrow{r} \widehat{\mathrm{Ext}}^*_G(C,M)[\pm|r|] \to \widehat{\mathrm{Ext}}^*_G(C, M/\!\!/r) \to \widehat{\mathrm{Ext}}^*_G(C,M)[\pm 1]$$

If r is not a zero divisor on $\widehat{\mathrm{Ext}}^*_G(C,M)$:

$$\widehat{\mathrm{Ext}}^*_G(C, M/\!\!/r) \cong \frac{\widehat{\mathrm{Ext}}^*_G(C,M)[\pm|r|]}{r\,\widehat{\mathrm{Ext}}_G(C,M)}$$

In general, all we can say is

$$\mathrm{supp}_G(M/\!\!/r) = \mathrm{supp}_G(M) \cap V(r)$$

Let $k \hookrightarrow K$ be a field extension.
$\Rightarrow$ get $kG \longrightarrow KG$ ring hom.

$$\Rightarrow R := \mathrm{Ext}^*_G(k,k) \xrightarrow{K \otimes_k -} \mathrm{Ext}^*_G(K,K) =: R_K$$

$\rightsquigarrow$ adjoints
$$\mathrm{St\,Mod}(kG) \underset{(\ )\downarrow_{kG}}{\overset{K \otimes_k -}{\rightleftarrows}} \mathrm{StMod}(KG)$$

$$\begin{array}{c} \mathrm{St\,Mod}(kG) \\ \rotatebox{90}{$\subseteq$} \\ \Gamma_{\mathfrak{p}}(\mathrm{StMod}\, kG) \\ \rotatebox{90}{$\in$} \\ \Gamma_{\mathfrak{p}}(k) = (\Gamma_{\mathcal{V}(\mathfrak{p})} k)_{\mathfrak{p}} \end{array}$$

what we want to understand

<u>Theorem</u>: Let K, m and $\underline{b}$ be as before. Then one has
$$\Gamma_{\mathfrak{p}}(k) \cong \Gamma_m(K /\!\!/ \underline{b})\downarrow_{kG} \quad \text{in StMod } kG$$
Therefore
$$\Gamma_{\mathfrak{p}}(M) \cong \Gamma_m(M_K \otimes_K K/\!\!/\underline{b})\downarrow_{kG}$$
$$= \Gamma_{\nu(m)}(M_K \otimes_K K/\!\!/\underline{b})$$
Moreover if M is $\mathfrak{p}$-torsion then
$$\Gamma_{\mathfrak{p}} M = M_{\mathfrak{p}} = M \otimes_k (K/\!\!/\underline{b})\downarrow_{kG}.$$

<u>Remark</u>: Say $\mathfrak{p} = (0)$. Then $m = (\underline{b})$, so
$$\Gamma_{\mathfrak{p}} k = (K /\!\!/ \underline{b})\downarrow_{kG}$$

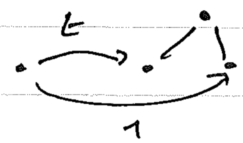
<u>Example</u>: $kV_4 = k[z_1, z_2]/(z_1^2, z_2^2)$, char $k = 2$.

$R = \mathrm{Ext}^*_{V_4}(k,k) = k[x_1, x_2]$, $|x_1| = 1 = |x_2|$.

$\mathfrak{p} = (0)$, $K = k(t)$, $m = (x_2 - tx_1)$

$\Gamma_{\mathfrak{p}} k = K /\!\!/ x_2 - tx_1$, $\quad K \xrightarrow{x_2 - tx_1} \Omega^{-1}K$

$\Omega^{-1}K =$ 



$$K/\!\!/_{x_2 - tx_1} = {}_1(\;)t$$

$$\Rightarrow \Gamma_{(0)} k = \begin{matrix} k(t) \\ {}_1(\quad)t \\ k(t) \end{matrix}$$

<u>Theorem</u>: $(\;)\downarrow_{kG}$ restricts to a functor
$$\Gamma_m(\text{StMod } KG) \longrightarrow \Gamma_{\mathfrak{p}}(\text{StMod } kG)$$
and this functor is full & <u>dense</u>.

essentially surjective

<u>Application</u>: $\Gamma_{\mathcal{V}(R)}(\mathrm{StMod}\ kG)$ are compactly generated.

- $\mathfrak{m} \in \mathrm{Proj}\ R$, a closed point
  $\Gamma_{\mathcal{V}(\mathfrak{m})}(\mathrm{StMod}\ kG)^c = \{M \in \mathrm{stMod}\ kG \mid \mathrm{supp}_G\, M \subseteq \{\mathfrak{m}\}\}$

- More generally, $\Gamma_{\mathcal{V}}(\mathrm{StMod}\ kG)^c$
  $= \{M \in \mathrm{stMod}\ kG \mid \mathrm{supp}_G(M) \subseteq \mathcal{V}\}$
  for $\mathcal{V} \subseteq \mathrm{Proj}\ R$ specialisation closed,

<u>Corollary</u>: The compact objects in $\Gamma_{\mathfrak{p}}(\mathrm{StMod}\ kG)$ are precisely the objects of the form $N\downarrow_{kG}$ where ~~$N \in \mathrm{stMod}\ kG$~~ $N \in \mathrm{stMod}\ KG$ and $\mathrm{supp}_G(N) \subseteq \{\mathfrak{m}\}$.