St Mod

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Lecture 3 (Henning)

Triangulated categories with central ring actions

<u>Guiding example</u>: StMod(kG) with action of $H^*(G;k)$.

$\mathcal{T}$ denotes a Δed category, suspension $\Sigma : \mathcal{T} \xrightarrow{\sim} \mathcal{T}$. Suppose $\mathcal{T}$ has (set-indexed) coproducts. $X \in \mathcal{T}$ is <u>compact</u> if $\hom_{\mathcal{T}}(X, -)$ preserves $\coprod$s.

$\mathcal{T}^c$ = full subcat of compact objects (a <u>thick</u> subcat of $\mathcal{T}$).
closed under extensions

Say $\mathcal{T}$ is <u>compactly generated</u> if
- $\mathcal{T}^c$ is <u>essentially small</u> ← equivalent to a small category.
- $\mathrm{Hom}(-, X)|_{\mathcal{T}^c} = 0 \Rightarrow X = 0$ for all $X \in \mathcal{T}$.

<u>Prop^n</u>: StMod(kG) is compactly generated and $\underline{\mathrm{stmod}}(kG) \xrightarrow{\sim} (\mathrm{StMod}(kG))^c$
f.g. kG-mods

$R = \bigoplus_{n \in \mathbb{Z}} R^n$ a graded Noeth. comm.^{n} ring.

For $X, Y \in \mathcal{T}$, $\mathrm{Hom}^*_{\mathcal{T}}(X, Y) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n Y)$
action
$\mathrm{End}^*(X)$ and $\mathrm{End}^*(Y)$

A <u>central action</u> of R on $\mathcal{T}$ is given by

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$\varphi_X : R \longrightarrow \mathrm{End}^*_{\mathcal{T}}(X)$ for each $X \in \mathcal{T}$
such that $\forall \alpha \in \mathrm{Hom}^*_{\mathcal{T}}(X, Y)$, $r \in R$ homog.
we have
$$\varphi_Y(r)\,\alpha = (-1)^{|r||\alpha|}\,\alpha\,\varphi_X(r)$$

<u>Propn</u>: $H^*(G,k)$ acts centrally on $\mathrm{StMod}(kG)$

⌈ In $\mathcal{T} = \mathrm{StMod}(kG)$,
$$\underline{\mathrm{Hom}}^*(X, Y) \cong \widehat{\mathrm{Ext}}^*_{kG}(X, Y)$$
which gives action of $H^*(G,k)$. We could take $\widehat{\mathrm{Ext}}_{kG}(k,k)$ instead but cleaner to work with $H^*(G,k)$ because the former is often not Noetherian ⌋

⌈ There is a canonical map
$$\mathrm{Ext}^*(X, Y) \longrightarrow \widehat{\mathrm{Ext}}^*(X,Y)$$
$$i(Y) \xmapsto{\text{induces}} t(Y)$$ ⌋

Localising subcategories

<u>Recall</u>: $\mathcal{S} \underset{\text{full}}{\subseteq} \mathcal{T}$ is <u>localising</u> if $\mathcal{S}$ is a Δed subcat closed under $\amalg$ (in $\mathcal{T}$)

<u>Aim</u>: Classify localising subcats of $\mathcal{T}$ via the action of R (i.e. $\mathrm{Spec}\, R$).

$\mathrm{Spec}\, R$ = set of <u>homog.</u> prime ideals of R.

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Recall: $\mathcal{V} \subseteq \operatorname{Spec} R$ is <u>specialisation closed</u> if $\mathfrak{p} \in \mathcal{V}$, $\mathfrak{p} \subseteq \mathfrak{q} \Rightarrow \mathfrak{q} \in \mathcal{V}$.

$$\mathcal{T}_{\mathcal{V}} := \{ X \in \mathcal{T} \mid \operatorname{Hom}^*(C,X) \text{ is } \underbrace{\mathcal{V}\text{-torsion}}_{\operatorname{Hom}^*(C,X)_{\mathfrak{q}} = 0 \ \forall \mathfrak{q} \notin \mathcal{V}} \text{ for all } C \in \mathcal{T}^c \}$$

which is a localising subcat.

For $\mathfrak{p} \in \operatorname{Spec} R$:

$$\mathcal{T}_{\mathfrak{p}} := \{ X \in \mathcal{T} \mid \operatorname{Hom}^*(C,X) \text{ is } \underbrace{\mathfrak{p}\text{-local}}_{\operatorname{Hom}^*(C,X) \xrightarrow{\sim} \operatorname{Hom}^*(C,X)_{\mathfrak{p}}} \text{ for all } C \in \mathcal{T}^c \}$$

which is also a loc. subcat.

<u>Localisation functors</u>

A (Bousfield) localisation functor is a pair (L, η):
$L : \mathcal{T} \to \mathcal{T}$ is an exact functor,
$\eta : \mathrm{id} \Rightarrow L$ nat. trans., satisfying
1) $L\eta : L \to L^2$ is an isom., and
2) $L\eta = \eta L$

[equivalently, $\exists$ an adjoint pair $\mathcal{T} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{U}$ of exact functors s.t. $L = G \circ F$ and $\eta =$ unit $\mathrm{Id} \to GF$, and G is fully faithful.

$$\mathcal{T} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{U}$$

$\mathcal{U}$ is the subcat of local objects.

<u>Example</u>: $Ab = \mathrm{Mod}\,\mathbb{Z} \underset{\longleftarrow}{\overset{F}{\longrightarrow}} \mathrm{Mod}\,\mathbb{Q}$

$\mathrm{Ker}\ F$ = Torsion groups.

$\rightsquigarrow\ D(\mathrm{Mod}\,\mathbb{Z}) \rightleftarrows D(\mathrm{Mod}\,\mathbb{Q})$ ┘

~~Theorem: For $\mathcal{V} \subseteq \mathrm{Spec}\,R$ specialisation closed~~

There is the dual notion of a colocalisation functor (Γ, ε).

<u>Lemma</u>: If (L, η) localisation functor then there is a colocal. functor (Γ, ε) for $\mathcal{T}$ and a functorial exact triangle

$$\Gamma X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} LX \longrightarrow$$

and $\mathrm{Image}(\Gamma) = \mathrm{Ker}(L)$ and
$\mathrm{Image}(L) = \mathrm{Ker}(\Gamma)$ forming an <u>orthogonal pair</u> w/r/t $\mathrm{Hom}^*(-,-)$
$(\mathrm{Hom}^*(\Gamma X, LX) = 0$

<u>Theorem</u>: For $\mathcal{V} \subseteq \mathrm{Spec}\,R$ spec. closed there is a colocalisation functor
$\Gamma_{\mathcal{V}} : \mathcal{T} \to \mathcal{T}$ with $\mathrm{Im}\,\Gamma_{\mathcal{V}} = \mathcal{T}_{\mathcal{V}}$.
For $\mathfrak{p} \in \mathrm{Spec}\,R$, set
$$Z(\mathfrak{p}) = \{\mathfrak{q} \in \mathrm{Spec}\,R \mid \mathfrak{q} \not\subseteq \mathfrak{p}\}$$
Then
$$L_{Z(\mathfrak{p})} : \mathcal{T} \to \mathcal{T};\ X \longmapsto X_{\mathfrak{p}}.$$
has image $\mathcal{T}_{\mathfrak{p}}$.

Moreover $X \longmapsto X_{\mathfrak{p}}$ induces an ~~map~~ isom.

$$\mathrm{Hom}^*(C,X)_{\mathfrak{p}} \xrightarrow{\sim} \mathrm{Hom}^*(C,X_{\mathfrak{p}})$$

Now define for $\mathfrak{p} \in \operatorname{Spec} R$

$$\Gamma_{\mathfrak{p}} X := (\Gamma_{\mathcal{V}(\mathfrak{p})} X)_{\mathfrak{p}} \cong \Gamma_{\mathcal{V}(\mathfrak{p})}(X_{\mathfrak{p}})$$

(a $\mathfrak{p}$-local and $\mathfrak{p}$-torsion object)
the <u>local cohomology functor</u>.

<u>Tensor Δed categories</u>

$(\mathcal{T}, \otimes, \mathbb{1})$ is a $\otimes$-Δed cat. if

- $\mathcal{T}$ is a compactly gen. Δed cat.
- $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ symmetric monoidal
- $\otimes$ is exact and preserves $\coprod$ in each variable
- $\mathbb{1}$ is a unit for $\otimes$ and compact.

$\mathrm{End}^*(\mathbb{1})$ is graded comm. and acts centrally on $\mathcal{T}$ via

$$\mathrm{End}^*(\mathbb{1}) \xrightarrow{-\otimes X} \mathrm{End}^*(X).$$

<u>Def^n</u>: Say R acts <u>canonically</u> if action of R is given by a ring hom
$R \longrightarrow \mathrm{End}^*(\mathbb{1})$

<u>Example</u>: $H^*(G;k) \longrightarrow \mathrm{Ext}^*(k,k)$ for
$\mathrm{StMod}\, kG$.

<u>Lemma</u>: If R acts canonically then

$$\Gamma_{\mathfrak{p}} X \cong (\Gamma_{\mathfrak{p}} \mathbb{1}) \otimes X$$
$$\Gamma_{\mathcal{V}} X \cong (\Gamma_{\mathcal{V}} \mathbb{1}) \otimes X$$
$$X_{\mathfrak{p}} \cong \mathbb{1}_{\mathfrak{p}} \otimes X.$$

Say $\mathcal{C} \subseteq \mathcal{T}$ is a <u>tensor ideal</u> if
$X \in \mathcal{C}, Y \in \mathcal{T} \Rightarrow X \otimes Y \in \mathcal{C}$.

For $\mathcal{C} \subseteq \mathcal{T}$ set $\mathrm{Loc}(\mathcal{C})$ = smallest loc. subcat. of $\mathcal{T}$ containing $\mathcal{C}$

$\mathrm{Loc}^{\otimes}(\mathcal{C}) :=$ tensor ideal loc. subcat. gen by $\mathcal{C}$.

<u>Thm</u> (Local-global principle):

$$\mathrm{Loc}^{\otimes}(X) = \mathrm{Loc}^{\otimes}\{\Gamma_{\mathfrak{p}}(X) \mid \mathfrak{p} \in \operatorname{Spec} R\}$$
$\forall X \in \mathcal{T}$.

<u>Def^n</u>: $\mathcal{T}$ is <u>stratified</u> by the action of R if for each $\mathfrak{p} \in \operatorname{Spec} R$ either
$\Gamma_{\mathfrak{p}} \mathcal{T} = \{0\}$ or $\Gamma_{\mathfrak{p}} \mathcal{T}$ is minimal (i.e. has no proper $\otimes$-ideal localising subcat.)

Note $\Gamma_{\mathfrak{p}} \mathcal{T}$ is always a $\otimes$-ideal loc. subcat.

Thm: Suppose $\mathcal{T}$ stratified by R. Then

$$\{\mathcal{S} \subseteq \mathcal{T} \text{ } \otimes\text{-ideal loc.}\} \xrightarrow{\sim} \{\text{subsets of } \mathrm{supp}_R(\mathcal{T})\}$$
$$\mathcal{S} \longmapsto \mathrm{supp}_R(\mathcal{S})$$

For $X \in \mathcal{T}$, $\mathrm{supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \Gamma_{\mathfrak{p}} X \neq 0\}$

For $\mathcal{C} \subseteq \mathcal{T}$, $\mathrm{supp}_R(\mathcal{C}) = \bigcup_{X \in \mathcal{C}} \mathrm{supp}_R(X)$.

The construction of $\Gamma_{\mathfrak{p}}$:

$\mathcal{V} \subseteq \mathrm{Spec}\, R$, $\mathcal{V}(I)$, $I \subseteq R$, I gen. by $r_1, \dots, r_n$.
$\Gamma_{\mathcal{V}} = \Gamma_{\mathcal{V}(r_1)} \otimes \dots \otimes \Gamma_{\mathcal{V}(r_n)}$

Want to compute $\Gamma_{\mathcal{V}(r)}$ for $r \in R$.
Say $|r| = d$. For $X \in \mathcal{T}$ get
$\mathrm{Cone}(X \xrightarrow{r^i} X_i := \Sigma^{di} X) =: X/\!\!/r^i$
for $i \geq 0$.

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "r"'] & X \arrow[r, equal] \arrow[d, "r^2"] & X \arrow[r] \arrow[d, "r^3"] & \cdots & & X \arrow[d, "\eta"] \\
X_1 \arrow[r, "r"] \arrow[d] & X_2 \arrow[r, "r"] \arrow[d] & X_3 \arrow[r] \arrow[d] & \cdots \arrow[r, squiggly, "\text{hocolim}"] & & L_{\mathcal{V}(r)} X \arrow[d] \\
X/\!\!/r \arrow[r] & X/\!\!/r^2 \arrow[r] & X/\!\!/r^3 \arrow[r] & \cdots & & \Sigma(\Gamma_{\mathcal{V}(r)} X)
\end{tikzcd}$$

⌈ What is hocolim in $\mathcal{T}$?

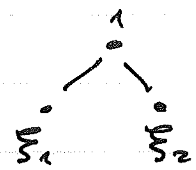
$$\coprod_{i\geq 0} X_i \xrightarrow{\text{id-shift}} \coprod_{i\geq 0} X_i \longrightarrow \operatorname{hocolim} X_i \longrightarrow$$ ⌋

<u>Klein 4-group</u>

We computed cohomology of $G = \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$ as

$$H^*(G,k) = k[\xi_1, \xi_2], \quad |\xi_i| = 1$$

where $\xi_i : k \longrightarrow \Omega^{-1}(k)$, i.e.



Compute $\Gamma_{\mathcal{V}(\xi_1)}(k)$: