Exercises for Tuesday

(1) Let T be a compactly generated triangulated category. Given any class C of compact objects, prove that there exists a localisation functor $L\colon \mathsf{T} \to \mathsf{T}$ such that $\operatorname{Ker} L = \mathsf{Loc}(\mathsf{C})$.

Hint: Use Brown representability to show that the inclusion $\mathsf{Loc}(\mathsf{C}) \to \mathsf{T}$ admits a right adjoint.

(2) Let $A = \left[\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right]$ be the algebra of 2×2 upper triangular matrices over a field k and let T denote the derived category of all A-modules. Up to isomorphism, there are precisely two indecomposable projective A-modules:

$$P_1 = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad P_2 = \begin{bmatrix} 0 & k \\ 0 & k \end{bmatrix}$$

satisfying $\operatorname{Hom}_A(P_1, P_2) \neq 0$ and $\operatorname{Hom}_A(P_2, P_1) = 0$. For $i = 1, 2$ let L_i denote the localisation functor such that the L_i-acyclic objects form the smallest localising subcategory containing P_i, viewed as a complex concentrated in degree zero. Show that $L_1 L_2 \neq L_2 L_1$.

(3) Let k be a field, $A = k[x]/x(x-1)$, and T its derived category $\mathsf{D}(A)$. Then T is A-linear, hence also k-linear, via restriction along the homomorphism $k \to A$. Prove that T has four localising subcategories. Hence T cannot be stratified by the k-action. It is however stratified by the A-action; this is a special case of Neeman's theorem, but can be verified directly.

(4) Prove that $\operatorname{supp}_R \mathsf{T} = \operatorname{supp}_R \mathsf{T}^{\mathsf{c}}$.

(5) Let $\mathfrak{p}$ be a homogeneous prime ideal in R. Prove that $\varGamma_{\mathfrak{p}} \cong \varGamma_{\mathcal{V}(\mathfrak{p})}$ when $\mathfrak{p}$ is maximal, with respect to inclusion, in $\operatorname{supp}_R \mathsf{T}$, and that $\varGamma_{\mathfrak{p}} \cong L_{\mathcal{Z}(\mathfrak{p})}$ when $\mathfrak{p}$ is minimal in $\operatorname{supp}_R \mathsf{T}$.