Exercises for Monday

In the following exercises k is a field, G a finite group, and M, N are kG-modules. Keep in mind that the kG action on $M \otimes_k N$ is via the diagonal. In what follows $\operatorname{char} k$ denotes the characteristic of k.

(1) Prove that the canonical homomorphisms $G \to G \times H$ and $H \to G \times H$ of groups induce an isomorphism of k-algebras:
$$kG \otimes_k kH \xrightarrow{\cong} k[G \times H]\,.$$
(2) Let $G = \langle g_1, \dots, g_r \rangle \cong (\mathbb{Z}/p)^r$ and set $x_i = g_i - 1$, in kG. Prove that if $\operatorname{char} k = p$, then kG is isomorphic as a k-algebra to
$$k[x_1, \dots, x_r]/(x_1^p, \dots, x_r^p)\,.$$
(3) Let $\pi\colon kG \to k$ be the k-linear map defined on the basis G by $\pi(1) = 1$ and $\pi(g) = 0$ for $g \neq 1$. Verify that the following map is a kG-linear isomorphism.
$$kG \to \operatorname{Hom}_k(kG, k) \quad \text{where } g \mapsto [h \mapsto \pi(g^{-1}h)]\,.$$
This proves that kG is a self-injective algebra.

(4) Verify that the following maps are kG-linear isomorphisms.
$$M \to k \otimes_k M \qquad \text{where } m \mapsto 1 \otimes m\,;$$
$$M{\downarrow_1}{\uparrow^G} \to kG \otimes_k M \quad \text{where } g \otimes m \mapsto g \otimes gm\,.$$
The second isomorphism implies that $kG \otimes_k M$ is a free kG-module.

(5) Verify that the following maps are kG-linear monomorphisms:
$$M \to kG \otimes_k M \quad \text{where } m \mapsto \sum_{g \in G} g \otimes m\,;$$
$$M \to M{\downarrow_1}{\uparrow^G} \qquad \text{where } m \mapsto \sum_{g \in G} g \otimes g^{-1}m\,.$$
Since $M{\downarrow_1}{\uparrow^G}$ is free, it follows that each module embeds into a free one, in a canonical way.

(6) Let H be a subgroup of G. Prove that kG is free as a kH-module, both on the left and on the right, and describe bases.

(7) Prove that a kG-module M is projective if and only if it is injective. Hint: use (3) and (5). It is also true that M is projective if and only if it is flat.

(8) Let G be a finite p-group and $\operatorname{char} k = p$. For any non-zero element $m \in M$ the $\mathbb{F}_p$-subspace of M spanned by $\{gm \mid g \in G\}$ is finite dimensional, and so has p^n elements for some n. Show that some non-zero element of this set is fixed by G. Deduce that the trivial module is the only simple kG-module.

(9) Let $G = (\mathbb{Z}/p)^r$ and $\operatorname{char} k = p$. Describe $J(kG)$ and show that $J(kG)/J^2(kG)$ is a vector space of dimension r over k. Prove that there is a natural isomorphism of k-vector spaces
$$H^1(G, k) \cong \operatorname{Hom}_k(J(kG)/J^2(kG), k)\,.$$
(10) Let $G = \langle g \mid g^{p^n} = 1 \rangle \cong \mathbb{Z}/p^n$ with $n > 1$ and $\operatorname{char} k = p$. Use Jordan canonical form to show that a finitely generated kG-module is free if and only if its restriction to the subgroup
$$H = \langle g^{p^{n-1}} \rangle \cong \mathbb{Z}/p$$
is free. This is a case of Chouinard's theorem.

(11) Compute $\operatorname{supp}_{\mathbb{Z}} M$ for a finitely generated abelian group M, and $\operatorname{supp}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z})$.