St Mod - Lecture 2 (Henning)

The (stable) module category of a finite group.

The setup:

G finite group $\rightsquigarrow kG$, the group algebra
k a field
Mod kG = category of kG-modules
(k-linear representations of G)

Problem: classify the representations of G. !

1) kG is semi-simple iff $p = \mathrm{char}\, k \nmid |G|$
(Maschke, 1899)

2) kG is of finite representation type
(i.e. there are only finitely many indecomposable representations up to $\cong$) iff p-Sylow subgroups of G are cyclic.
(Jannsz, '66, Kupusch, '69)

From now on, assume $p := \mathrm{char}\, k \mid |G|$.
mod kG := finite dim kG-mods.

Note: The prime interest is in mod kG, but our techniques & results also involve infinite representations.

Have seen $D(\mathrm{Mod}\ A)$, for A comm. Noeth., earlier.

$$\begin{array}{ccc} D(\mathrm{Mod}\ A) & \longleftrightarrow & \mathrm{St\,Mod}\ kG \\ \circlearrowleft & & \circlearrowleft \\ A & & H^*(G,k) \end{array}$$

$\ulcorner$ $\mathcal{T}$ triangulated category
$\circlearrowleft$ with action of
R comm. Noeth. ring $\lrcorner$

<u>Group algebra</u>:

$kG = \bigoplus_{g\in G} kg$ fin. dim. k-algebra.

~~Examples~~:

G is abelian iff kG commutative.

<u>Examples</u>:

1) $G = \langle g \mid g^d = 1\rangle$
$kG = k[x]/(x^d-1)$

2) $G = \left(\frac{\mathbb{Z}}{p}\right)^r = \underbrace{\frac{\mathbb{Z}}{p} \times \dots \times \frac{\mathbb{Z}}{p}}_{r}$

(an <u>elementary abelian p-group of rank r</u>)
char $k = p$, G gen. by $g_1, g_2, \dots g_r$
Let $z_i := g_i - 1$.
$\rightsquigarrow k[G] \cong k[z_1, \dots, z_r]/(z_1^p, \dots, z_r^p)$

Stable category / triangulated structure

$M, N \in \operatorname{Mod} kG$. ~~$\operatorname{Hom}_{kG}$~~

$\Rightarrow \underline{\operatorname{Hom}}_{kG}(M,N) := \dfrac{\operatorname{Hom}_{kG}(M,N)}{\{M \to P \to N \mid P \text{ projective}\}}$

$\operatorname{StMod} kG$ has objects kG-modules & morphisms are the <u>stable morphisms</u> $\underline{\operatorname{Hom}}_{kG}(-,-)$.

<u>Proposition</u>: For a kG-module M, TFAE:

(1) M is projective

(2) M is injective.

<u>Proof (sketch)</u>:

- Observe kG is an injective kG-module, even an injective <u>cogenerator</u> (i.e. any M embeds into $\prod_I kG$)
- ~~Injective~~ $\operatorname{Inj} kG :=$ full subcat of injective kG-mods $= \operatorname{Add} kG$, since kG Noeth.

  (for a module M, $\operatorname{Add} M :=$ ~~arbitrary direct sums~~ direct summands of arbitrary direct sums of copies of M)
- $\operatorname{Proj} kG =$ full subcat of proj. kG-modules $= \operatorname{Add} kG$. $\square$

<u>Corollary</u>: Mod kG is a <u>Frobenius category</u> (enough projs, enough injs, and projs = injs) and therefore St Mod kG is triangulated.

- The suspension is given by $M \mapsto \Omega^{-1}M := \operatorname{coker}(M \hookrightarrow E(M))$
- The exact sequences $0 \to L \to M \to N \to 0$ in Mod kG $\rightsquigarrow$ exact triangles

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \Omega^{-1}L$$

where γ comes from completing

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, equal] & M \arrow[r, "\beta"] \arrow[d] & N \arrow[r] \arrow[d, dashed, "\gamma"] & 0 \\ 0 \arrow[r] & L \arrow[r] \arrow[ur, dashed] & E(L) \arrow[r] & \Omega^{-1}L \arrow[r] & {} \end{tikzcd}$$

exists since $E(L)$ injective

One of the axioms: $L \xrightarrow{\alpha} M$ gives exact triangle $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} \Omega^{-1}L$.

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\alpha"'] & E(L) \arrow[r] \arrow[d] & \Omega^{-1}(L) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M \arrow[r, "\beta"'] \arrow[ur, phantom, "\ulcorner"] & N \arrow[r, "\gamma"'] & \Omega^{-1}(L) \arrow[r] & 0 \end{tikzcd}$$

St Mod kG has arbitrary coproducts

<u>Prop</u>: St Mod kG is a compactly generated triangulated category.

Tensor structure

If M, N are kG-mods, can form $M \otimes_k N \in \mathrm{Mod}_{kG}$ (with diagonal G-action), i.e. $g(m \otimes n) = gm \otimes gn$)

$\mathrm{Hom}_k(M, N) \in \mathrm{Mod}_{kG}$ (with diagonal G-action, $(g \cdot \alpha)(m) = g\alpha(g^{-1}m)$)

$\rightsquigarrow$ tensor structure on $\mathrm{StMod}\, kG$.

($\otimes_k$ is exact, as is $\hom_k(-,-)$, so we don't need to derive. We also use that $M \otimes_k P$ is projective when P is)

$M^G := \{m \in M \mid gm = m \ \forall g \in G\}$, the <u>invariant submodule</u>

k = trivial representation.

$\mathrm{Hom}_{kG}(k, M) = M^G$, and $\mathrm{Hom}_k(M, N)^G = \mathrm{Hom}_{kG}(M, N)$

Cohomology:

$H^*(G; M) := \mathrm{Ext}^*_{kG}(k, M)$
(deriving $M \mapsto M^G$)

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Cup product

$$H^i(G,M) \underset{k}{\otimes} H^j(G,N) \longrightarrow H^{i+j}(G, M \underset{k}{\otimes} N)$$

Yoneda product

$$\mathrm{Ext}^i_{kG}(M,N) \otimes \mathrm{Ext}^j_{kG}(L,M) \longrightarrow \mathrm{Ext}^{i+j}_{kG}(L,N)$$

These coincide when $L = M = N = k$.

<u>Prop^n</u>: $H^*(G,k)$ is a graded commutative k-algebra

$\ulcorner$ graded commutative $\rightsquigarrow$ $x \cdot y = (-1)^{|x||y|} y \cdot x$ $\lrcorner$

<u>Example</u>: $G = (\mathbb{Z}/2)^r$, $p = 2$, then
$H^*(G,k) \cong k[\xi_1, \dots, \xi_r]$, $|\xi_i| = 1$

<u>Thm</u> (Golod – Evens – Venkov ~ 1960)
$H^*(G,k)$ is Noetherian.

<u>Tate cohomology</u> $M, N \in \mathrm{Mod}\, kG$.

$$\widehat{\mathrm{Ext}}^*_{kG}(M,N) := H^*(\mathrm{Hom}_{kG}(M, t(N)))$$

$$\underbrace{\underbrace{\cdots \to I^{-2} \to I^{-1}}_{\text{projective resolution}} \searrow \; N \; \nearrow \underbrace{I^0 \to I^1 \to \cdots}_{\text{injective coresolution}}}_{\text{Tate resolution } t(N).}$$

<u>Lemma</u>: $\widehat{\operatorname{Ext}}^*_{kG}(M,N) \cong \bigoplus_{i\in\mathbb{Z}} \underline{\operatorname{Hom}}(M, \Omega^{-i}(N))$

Next time: we'll see $\widehat{\operatorname{Ext}}^*(M,N)$ is an $H^*(G,k)$-module $\rightsquigarrow$ action of $H^*(G,k)$ on $\operatorname{StMod} kG$.

<u>Example</u>: Klein 4-group $G = \langle g_1, g_2 \rangle \cong \frac{\mathbb{Z}}{2} \times \frac{\mathbb{Z}}{2}$.

$x_i := g_i - 1 \rightsquigarrow x_i^2 = 0$ and
$kG \cong k[x_1, x_2]/(x_1^2, x_2^2)$

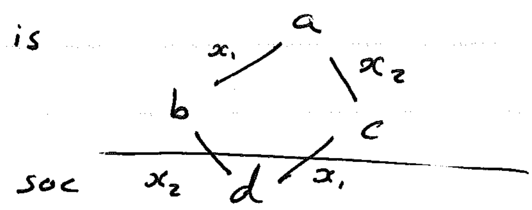
Describe kG-modules by diagrams

vertices $\longleftrightarrow$ base elements

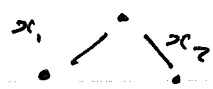
$\begin{array}{c} a \\ \mid x_i \\ b \end{array} \longleftrightarrow x_i \cdot a = b$

no edge $\begin{array}{c} a \\ \vdots \\ b \end{array} \longleftrightarrow x_i \cdot a = 0$.

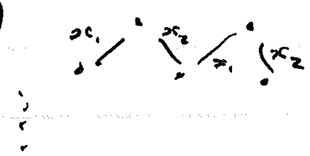
<u>Example</u>: kG is



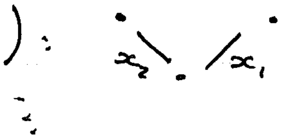
Syzygies of k: $\Omega^{-1}(k)$



$\Omega^{-2}(k)$



$\Omega^{1}(k)$:



For $n \geq 0$, $\mathrm{Ext}^n(k,k) \cong \underline{\mathrm{Hom}}(k, \Omega^{-n}(k))$
of dim^n $n+1$.

$\rightsquigarrow \mathrm{Ext}^*(k,k) \cong k[\xi_1, \xi_2]$, $|\xi_i| = 1$.

List of all fin. dim. indec. kG-modules, k alg. closed

- kG
- $\Omega^n(k)$, $k \in \mathbb{Z}$
- L_{r^n}, $(r,n) \in \mathbb{P}^1(k) \times \mathbb{N}_{\geq 1}$

$H^1(G,k) \ni r = r_1 \xi_1 + r_2 \xi_2 \quad (r_i \in k)$

$$0 \to k \xrightarrow{r^n} \Omega^{-n}(k) \longrightarrow L_{r^n} \to 0.$$

Further explanation:

$G = \mathbb{Z}/2 \times \mathbb{Z}/2$

$kG = k[x_1, x_2]/(x_1^2, x_2^2)$

M a fin. gen. kG-mod
$\rightsquigarrow$ basis $\mathcal{B}$, and $x_1 b$, $x_2 b$ for $b \in \mathcal{B}$.

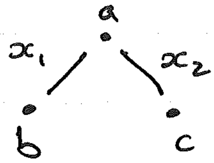
kG:
$$\begin{tikzcd}
& a \arrow[dl, dash, "x_1"'] \arrow[dr, dash, "x_2"] & \\
b \arrow[dr, dash, "x_2"'] & & c \arrow[dl, dash, "x_1"] \\
& d &
\end{tikzcd}$$

$x_1 a = b \qquad x_2 a = c$
$x_1 b = 0 \qquad x_2 b = d$
$x_1 c = d \qquad x_2 c = 0$
$x_1 d = x_2 d = 0.$

Can compute from these diagrams, e.g.
$\mathrm{Hom}_{kG}(k, kG) \cong kd$.

$\Rightarrow 0 \to k \to kG \to \Omega^{-1}(k) \to 0$
$\quad 1 \mapsto d$

so diagram for $\Omega^{-1}(k)$ is



Iterate; $\mathrm{Hom}_{kG}(k, \Omega^{-1}(k)) \cong kb \oplus kc$

$0 \to \Omega^{-1}(k) \to kG \oplus kG \to \Omega^{-2}(k)$