StMod - Lecture 1 (Srikanth)

① A a commutative Noetherian ring (e.g. $\mathbb{Z}$, $k[x_1, \ldots, x_n]$ & their quotients)

$\operatorname{Spec} A :=$ collection of prime ideals in A.

For $I \leq A$ ideal, define
$$V(I) = \{ \mathfrak{p} \in \operatorname{Spec} A \mid \mathfrak{p} \supseteq I \}$$

- These form the closed subsets of the <u>Zariski topology</u> on $\operatorname{Spec} A$.

<u>Defn</u>: $V \subseteq \operatorname{Spec} A$ is said to be <u>specialisation closed</u> if $\mathfrak{p} \in V$ and $\mathfrak{q} \supseteq \mathfrak{p} \Rightarrow \mathfrak{q} \in V$

$\in \operatorname{Spec} A$

$$\Leftrightarrow \mathfrak{p} \in V \Rightarrow V(\mathfrak{p}) \subseteq V$$
$$\Leftrightarrow V = \bigcup_{\lambda} V(I_\lambda)$$

Given $\mathcal{U} \subseteq \operatorname{Spec} A$, $cl(\mathcal{U}) := \bigcup_{\mathfrak{p} \in \mathcal{U}} V(\mathfrak{p})$

~~Notation~~ $\operatorname{mod} A$ = f.g. A-modules
$\cap$
$\operatorname{Mod} A$ = A-modules.

<u>Def^n</u>: $M \in \operatorname{Mod} A$, $\operatorname{Ass}_A M := \{ \mathfrak{p} \in \operatorname{Spec} A \mid \frac{A}{\mathfrak{p}} \hookrightarrow M \}$

If $M \neq 0$ then $\mathrm{Ass}_R M \neq \emptyset$. This is a consequence of:

<u>Prop</u>: The collection of ideals
$$\{ I \leq A \mid A/I \hookrightarrow M \}$$
has maximal elements (w.r.t. $\leq$) and each max. element is prime.

<u>Proof</u>: Say I is maximal. $A/I \hookrightarrow M$ means $\exists x \in M$ s.t. $\mathrm{ann}_A x = I$

$$\mathrm{ann}_A x = \{a \in A \mid a \cdot x = 0\}$$

(so collection $\neq \emptyset$)

Given $a, b \notin I$. Then $\mathrm{ann}_A(bx) \underset{=}{\supseteq} \mathrm{ann}_A(x) = I$
$\underset{=}{\subseteq} \not\subset \mathrm{ann}_A(ax)$

$\therefore (ab)\cdot x \neq 0$ (else $a(bx) = 0 \Rightarrow a \in \mathrm{ann}_A(x)$)
$\therefore ab \notin I$ and I is prime $\square$.

<u>Properties</u>:

① $\mathfrak{p} \in \mathrm{Ass}\, M \iff \mathrm{Hom}_{A_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}) \neq 0$

where $k(\mathfrak{p}) = (A/\mathfrak{p})_\mathfrak{p}$, the residue field at $\mathfrak{p}$.

② $\mathrm{Ass}_A(A/\mathfrak{p}) = \{\mathfrak{p}\}$

③ $0 \to M' \to M \to M'' \to 0$ exact
$\Rightarrow \mathrm{Ass}_A(M') \subseteq \mathrm{Ass}_A(M) \subseteq \mathrm{Ass}_A(M') \cup \mathrm{Ass}_A(M'')$

④ If $M \in \text{mod } A$, then $|\text{Ass } M| < \infty$.

<u>Corollary</u>: (of prop^n)

$M \in \text{mod } A \Rightarrow \exists$ filtration $0 \leq M_0 \leq \dots \leq M_r = M$ s.t. $\frac{M_{i+1}}{M_i} \cong \frac{A}{\mathfrak{p}_i}$, $\mathfrak{p}_i \in \text{Spec } A$.

<u>Def^n</u>: $M \in \text{Mod } A$. The <u>big support</u> of M is:

$$\text{Supp}_A M = \{\mathfrak{p} \in \text{Spec } A \mid M_{\mathfrak{p}} \neq 0\}$$

Clearly $\text{Ass}_A M \subseteq \text{Supp}_A M$. $\text{Supp}_A M$ is always specialisation closed.

<u>Prop</u>: ① $\text{Supp}_A\left(\frac{A}{I}\right) = V(I)$

② $0 \to M' \to M \to M'' \to 0$ exact then $\text{Supp } M = \text{Supp } M' \cup \text{Supp } M''$.

③ $\text{Supp}_A\left(\bigoplus M_\lambda\right) = \bigcup \text{supp}(M_\lambda)$

(also true for Ass, gives a way of constructing non-f.g. mod with finitely many ass primes, e.g. $\bigoplus_{\mathbb{N}} A/\mathfrak{p}$

④ $\text{Supp}_A M \subseteq V(\text{ann}_A M)$ with equality when $M \in$ ~~Mod~~$_A$ mod_A

$[\text{ann}_A M = \ker(A \to \text{End}_A M)]$

Sketch of ④:

If $M = \sum_\lambda M_\lambda$ with $M_\lambda \cong \frac{A}{I_\lambda}$ then

$$\mathrm{Supp}_A M = \bigcup_\lambda \mathrm{Supp}_A M_\lambda = \bigcup_\lambda V(I_\lambda) \subseteq \underbrace{V(\bigcap_\lambda I_\lambda)}_{\mathrm{ann}_A M}$$

equality holds ~~when~~ if sum is finite.

(n.b. since $\sum_\lambda$, not $\bigoplus_\lambda$ we need that localisation commutes w/ $\sum_\lambda$)

ⓒ $C \subseteq \mathrm{Mod}\, A$ is Serre if given $0 \to M' \to M \to M'' \to 0$ then $M', M'' \in C \iff M \in C$. C is localising if furthermore C is closed under ~~$\bigoplus$~~ arbitrary $\bigoplus$.

Def^n: $\mathrm{Supp}_A C = \bigcup_{M \in C} \mathrm{Supp}_A M$, specialisation closed.

Thm (Gabriel):

There are bijections

$$\left\{ \begin{array}{l} \text{Serre subcats} \\ \text{of mod } A \end{array} \right\} \xrightarrow{\mathrm{Supp}_A(-)} \left\{ \begin{array}{l} \text{specialisation-closed} \\ \text{subsets of Spec } A \end{array} \right\}$$

$$\{M \in \mathrm{mod}\, A \mid \mathrm{Supp}_A M \subseteq C\} \longleftarrow C$$

Similarly

$$\left\{ \begin{array}{l} \text{localising subcats} \\ \text{of Mod } A \end{array} \right\} \xrightarrow{\mathrm{Supp}_A(-)} \{ \quad '' \quad \}$$

ⓓ $\underset{\text{unbounded}}{D}(\mathrm{Mod}\ A) \ni M$, define <u>small support</u> or <u>support</u> of M

$$\mathrm{supp}_A(M) := \{ \mathfrak{p} \in \mathrm{Spec}\ A \mid M_{\mathfrak{p}} \overset{L}{\underset{A_{\mathfrak{p}}}{\otimes}} k(\mathfrak{p}) \neq 0 \}$$

$$M_{\mathfrak{p}} \overset{L}{\underset{A_{\mathfrak{p}}}{\otimes}} k(\mathfrak{p}) = M \overset{L}{\underset{A_{\mathfrak{p}}}{\otimes}} k(\mathfrak{p})$$

(Def^{n} of Foxby)

~~All Supps up to now have been supps!~~)

<u>Fact</u>: ① $M \neq 0 \Rightarrow \mathrm{supp}_A(M) \neq \emptyset$

② $M \in \mathrm{Mod}\ A$ implies

$$\mathrm{Ass}_A M \subseteq \mathrm{supp}_A M \underset{\substack{\uparrow \\ \text{can be strict}}}{\subseteq} \mathrm{Supp}_A M = \mathrm{cl}(\mathrm{supp}_A M)$$

(<u>Example</u>: $\mathrm{supp}_A(k(\mathfrak{p})) = \{\mathfrak{p}\}$

$\mathrm{Supp}_A(k(\mathfrak{p})) = V(\mathfrak{p})$.)

③ $\mathfrak{p} \in \mathrm{supp}_A M \Leftrightarrow \mathrm{RHom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, k(\mathfrak{p})) \neq 0$ (Foxby)

i.e. derived hom

<u>Thm (Neeman 1992)</u>:

$$\left\{ \begin{array}{l} \text{Localising subcats} \\ \text{of } D(\mathrm{Mod}_A) \end{array} \right\} \xrightarrow{\mathrm{supp}_A(-)} \left\{ \begin{array}{l} \text{arbitrary subset of} \\ \text{Spec } A \end{array} \right\}$$

(E) Fix $\mathfrak{p} \in \operatorname{Spec} A$, $M \in \operatorname{Mod} A$ is <u>$\mathfrak{p}$-local</u> if $M \xrightarrow{\simeq} M_{\mathfrak{p}}$

$$\Leftrightarrow \operatorname{supp}_A M \subseteq \{ q \in \operatorname{Spec} A \mid q \subseteq \mathfrak{p} \}$$

$$\updownarrow$$

$$\operatorname{Spec} A_{\mathfrak{p}}$$

M is <u>$\mathfrak{p}$-torsion</u> if $\forall x \in M$, $\mathfrak{p}^n \cdot x = 0$ for some $n \gg 0$.

$$\Leftrightarrow \mathfrak{p} \subseteq q \quad \forall q \in \operatorname{Ass}_A M$$

$$\Leftrightarrow \operatorname{supp}_A M \subseteq V(\mathfrak{p})$$

$$\Leftrightarrow \operatorname{Supp}_A M \subseteq V(\mathfrak{p}) \quad \text{i.e. } M_q = 0 \ \forall q \notin V(\mathfrak{p})$$

(F) <u>Injective modules</u>

$M \in \operatorname{Mod} A$, write $E_A(M)$ for injective hull of M

<u>Thm</u> (Matlis)

1. Arbitrary $\oplus$ of injectives is injective (recall: A Noetherian)
2. Each injective module decomposes as a direct sum of indecomposable injectives
3. $E(A/\mathfrak{p})$ is indecomposable injective, and each indecomposable injective is of this form.

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⌈Recall: $M \hookrightarrow E_A(M)$ ← injective
⌊essential defines the injective hull⌋

<u>Fact</u>: • $E(\frac{A}{\mathfrak{p}})$ is $\mathfrak{p}$-local & $\mathfrak{p}$-torsion

$$E\left(\frac{A}{\mathfrak{p}}\right)_{\mathfrak{q}} = \begin{cases} 0 & \mathfrak{p} \neq \mathfrak{q} \\ E\left(\frac{A}{\mathfrak{p}}\right) & \mathfrak{p} = \mathfrak{q} \end{cases}$$

Let I injective. Then

$$I \cong \bigoplus_{\mathfrak{p} \in \mathrm{Ass}\, I} E(A/\mathfrak{p})^{u(\mathfrak{p})}$$

where $u(\mathfrak{p}) = \mathrm{rank}_{k(\mathfrak{p})}(\mathrm{Hom}_{A_{\mathfrak{p}}}(k(\mathfrak{p}), I_{\mathfrak{p}}))$

Matlis
<u>~~Mathis~~ duality</u>: $(A, \mathfrak{m})$ local, i.e.
A noeth, comm., with unique max ideal ~~m~~ $\mathfrak{m}$.

$$\text{Let } (-)^\vee := \mathrm{Hom}_A(-, E(A/\mathfrak{m}))$$

Then $\left\{\begin{matrix}\text{finite length} \\ A\text{-mods}\end{matrix}\right\} \underset{(-)^\vee}{\overset{(-)^\vee}{\rightleftarrows}} \left\{\begin{matrix}\text{finite length} \\ A\text{-mods}\end{matrix}\right\}$

⌈<u>Example</u>: $k \hookrightarrow A$, $\mathrm{rank}_k A < \infty$ $\quad \frac{k[x]}{(x^n)}$

$$E(A/\mathfrak{m}) = \mathrm{Hom}_k(A, k)$$

$$(-)^\vee = \mathrm{Hom}_k(-, k)$$ ⌋

In particular, $M \xrightarrow{\cong} M^{\vee\vee}$