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Morrow I

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Fix prime number p for whole course.

Introduction to p-adic Hodge theory:

Goals: Indicate goals and objects of p-adic Hodge theory:

- Introduce main objects of BMS I: "Integral p-adic Hodge theory"
- Mention rel. to THH in BMS II: "Integral p-adic Hodge th. and THH"

Introduction:

K: complete discrete valuation field of char. 0 with perf. residue field k of char. p.

$\mathcal{O}_K$: ring of integers

X: proper sm. sch. $/ \mathcal{O}_K$.

$$\begin{tikzcd} & X \arrow[dl, rightsquigarrow] \arrow[dr, rightsquigarrow] & \\ \text{special fiber} & & \text{generic fiber} \end{tikzcd}$$

special fiber: X_k sm. proper over k; also $X_{\bar{k}}$

generic fiber: X_K sm. proper over K; and geom. gen. fiber $X_{\bar{K}} = X_K \times_K \bar{K}$

In ℓ-adic coh., the proper and sm. basechange theorems gives Isom.

$$H^n_{et}(X_{\bar{k}}, \mathbb{Z}_\ell) \cong H^n_{et}(X_{\bar{K}}, \mathbb{Z}_\ell)$$

for any prime $\ell \neq p$ (compatible with Galois actions). False if $\ell = p$!

Grothendieck (70s): Should replace $H^n_{et}(X_{\bar{k}}, \mathbb{Z}_p)$ by "crystalline coh." $H^n_{crys}(X_k / W(k))$ and compare

$$H^n_{crys}(X_k / W(k)) \longleftrightarrow H^n_{et}(X_{\bar{K}}, \mathbb{Z}_p).$$

Note: LHS related to diff. forms; RHS replacement for Betti coh. So Grothendieck asking for alg. analogue of (M sm. mfd. / $\mathbb{C}$)

$$\begin{array}{ccc} H^n_{dR}(M/\mathbb{C}) & \cong & H^n_B(M, \mathbb{C}) \\ \uparrow\sim & & \uparrow\sim \\ H^n_{dR}(M/\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} & & H^n_B(M, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

Precise answer proposed by Fontaine after inverting p:

Thm (Crystalline conj. – Fontaine, Messing, Bloch, Kato, Tsuji, Niziol) There is a natural isomorphism

$$H^n_{crys}(X_k/W(k)) \otimes_{W(k)} \mathbb{B}_{crys}$$
$$\cong H^n_{et}(X_{\bar{K}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{crys}$$

compatible with Galois actions etc. /

($\mathbb{B}_{crys}$ is Fontaine's crys. p-adic period ring; a big $\mathbb{Q}_p$-algebra, containing $W(k)$.)

Why care about this? Easy cor. (if you understand $\mathbb{B}_{crys}$) is:

$$\left(H^n_{et}(X_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{crys}\right)^{Gal(\bar{K}/K)}$$
$$\cong H^n_{crys}(X_k/W(k))[\tfrac{1}{p}]$$

i.e. coh. of $X_{\bar{K}}$ + Galois action gives coh. of X_k

Problem: Since $1/p \in \mathbb{B}_{crys}$, we only see rational cohomology.

Integral p-adic Hodge theory: Do this without inverting p.

Main idea of BMS I : We define a new coh. th. $H^n_{A_{\inf}}(X)$ that interpolates

$$H^n_{crys}(X_k/W(k)),\ H^n_{et}(X_{\bar{K}}, \mathbb{Z}_p),$$

$$\text{and } H^n_{dR}(X/\mathcal{O}_K).$$

This coh. th. is the hypercoh. of a certain cx. of presheaves in the Zariski topology on $X \hat{\times}_{\mathcal{O}_K} \mathcal{O}_{\bar{K}}$:

$$X \times_{\mathcal{O}_K} \mathcal{O}_{\bar{K}} \supset \mathrm{Spec}(R) \quad (R/\mathcal{O}_K \text{ sm.})$$

$$\longmapsto A\Omega_{\hat{R}}$$

where $A\Omega_{\hat{R}}$ is a certain cx. of $A_{\inf}$-modules, depending only on the p-adic completion $\hat{R}$ and where $A_{\inf}$ is Fontaine's infinitesimal period ring

$$A_{\inf} = \lim_F W_r(\hat{\mathcal{O}}_{\bar{K}}).$$

Main goal of course : Define this $\mathbb{E}_\infty$-algebra $A\Omega_{\hat{R}}$.

Rel. to THH: For $R/\mathcal{O}_K$ sm., we prove:

Thm (BMS II) There exists a natural descending filtration on $TP(R)^\wedge_p$ such that

$$\mathrm{gr}^i\, TP(R)^\wedge_p \simeq A\Omega_{\hat{R}}[2i],$$

compatible with $A_{\inf}$-module str. via $\pi_0(TP(\mathcal{O}_K)^\wedge_p) = A_{\inf}$.

Outline of course:
- Décalage functor.
- DR cx. and Cartier isom.
- DRW cx. and Cartier isom.
- Constr. of $A\Omega_{\hat{R}}$.
- Appl. to p-adic Hodge th.
- Rel. to THH.
- THH of smooth alg. / k.

Décalage functor: Modifying torsion. (Deligne, Berthelot-Ogus, Mazur)

Fix ring A, non-zero divisor $f \in A$.

<u>Def</u> If C is a cx. of A-mod. s.t.

(1) $C^n = 0$ for $n < 0$

(2) C^n is f-tors. free for $n \geq 0$,

then define subcomplex

$$\eta_f C \subset C$$

by

$$(\eta_f C)^n = \{x \in f^n C^n \mid dx \in f^{n+1} C^{n+1}\}. \quad //$$

Note: $H^n(\eta_f C) \to H^n(C)$ has kernel and cokernel killed by f^n.

<u>Lemma</u> The map $C^n \to (\eta_f C)^n$ given by $x \mapsto f^n x$ induces isom.

$$H^n(C)/H^n(C)[f] \xrightarrow{\sim} H^n(\eta_f C),$$

where $(-)[f]$ is sub-A-module annihilated by f. $-$

<u>Cor</u> If $C \to C'$ is a q.is. then so is $\eta_f C \to \eta_f C'$.

<u>Cor</u> For every cx. D of A-mod. s.t.

(1') $H^n(D) = 0$ for $n < 0$

(2') $H^0(D)$ is f-tors. free

then may define derived functor

$$\mathbb{L}\eta_f D := \eta_f C$$

where C satisfies (1) and (2) and $C \xrightarrow{\sim} D$.