Nordic Mathematical Team Contest 2012

Competition time. March 16, 12.00 CET – March 20, 18.00 CET.

Solutions. Solutions must be written in English. Preferred format is a pdf document compiled from a LaTeX source. Scanned solutions written by hand are acceptable, but the jury is at a liberty to deduct points for illegibility.

Submission. Solutions should be submitted to the chairman of the jury at:

qimh@math.uu.se

Questions. Questions regarding the formulation of the problems may be directed towards the chairman of the jury. Answers will be posted on the official website.

Score. Each problem is worth 6 points.

Problem 1. A Swedish and a Danish cow are tethered in a pasture. The Danish cow is chained to a pole in the middle of the field, whereas the Swedish cow is chained to the exterior of a circular barn (called Norway by some) with a rope just long enough for it to reach the diametrically opposite point. The rope constraining the Swedish cow, however, is 10% longer than that of its colleague.

Which cow has the greater area to graze?

Problem 2. For a positive integer n, define $\gamma(n)$ to be the number of ways to write n as the difference of two squares of natural numbers.

(a) Show that $\gamma(n)$ is always finite. (1 point)

(b) Find a formula for γ. The formula may implement any elementary and well-known number-theoretic functions. It may also be split up into several (finitely many!) cases. (5 points)

Problem 3. Calculate the sum

$$\frac{1}{1}+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\dots.$$

Problem 4. Define the *veridative* of a function $f\colon \mathbf{R} \to \mathbf{R}$ as

$$f^*(x) = \lim_{k \to 1} \frac{\ln \frac{f(kx)}{f(x)}}{\ln k}.$$

Solve the *fiderential equation*

$$f^*(x) = 2012 + x$$

(in continuously fiderentiable functions f).

Problem 5. Let $f\colon \mathbf{R}_+ \to \mathbf{R}_+$ be non-increasing and locally integrable. Prove that

$$\inf_{g \in L^\infty(\mathbf{R}_+)} \left(\sup_{t>0} |g(t)| + \int_0^\infty |f(t) - g(t)|\, dt \right) = \int_0^1 f(t)\, dt.$$

Is the infimum reached for some $g \in L^\infty(\mathbf{R}_+)$? Why?

Problem 6. The entire plane has been partitioned into pairwise congruent equilateral triangles by drawing straight lines. Does the set of lattice points contain the vertices of some regular 2012-gon?

Problem 7. All the roots of the real equation

$$x^n + nx^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0 = 0$$

are real and negative. Prove that $a_k \leqslant \binom{n}{k}$ for all k.

Problem 8. Define the sequence (a_n) by the initial values

$$\begin{cases} a_1 = 1 + 2012^{1/1} \\ a_2 = 1 + 2012^{1/2} \\ a_3 = 1 + 2012^{1/3} \\ a_4 = 1 + 2012^{1/4} \\ \vdots \\ a_{2012} = 1 + 2012^{1/2012} \end{cases}$$

and then recursively by

$$a_{2013+n} = \sum_{k=1}^{2012} a_{n+k}^{2011/2012}, \text{ for } n \geqslant 0.$$

Show that the sequence (a_n) converges and determine its limit.

Problem 9. For integrable functions $f, g\colon \mathbf{R}^d \to \mathbf{R}$ we define the convolution $f * g : \mathbf{R}^d \to \mathbf{R}$ by

$$f * g(x) = \int_{\mathbf{R}^n} f(y - x)g(y)\,dx.$$

For f as above, we define a scaled version $f_s : \mathbf{R}^d \to \mathbf{R}$ by $f_s(x) = 2^{ds} f(2^s x)$.

Define $h(x) = (1 + |x|)^{-d-1}$. Show that there exists a constant $C > 0$ such that

$$\tfrac{1}{C} h_p \leqslant h_p * h_q \leqslant C h_p$$

for all $q \geqslant p > 0$.

Problem 10.

(a) Prove that every computable (recursive) sequence of (natural) numbers has either

- a strictly concave infinite subsequence,
- or a linear infinite subsequence,
- or a strictly convex infinite subsequence.

(3 points)

(b) Does there exist a computable function which, given a computable sequence (in the form of a program or Turing machine), computes the *values* of such an infinite subsequence? (2 points)

(c) Does there exist a computable function which, given a computable sequence, computes the (strictly increasing) sequence of *indices* of such an infinite subsequence? (1 point)

For example, the sequence

$$3, 7, 5, 6, 9, 2, 9, 11$$

has a linear subsequence

$$3, \cdot, 5, 6, \cdot, \cdot, 9, \cdot.$$

(If you like, concavity, linearity, and convexity refers to the piecewise linear graph obtained by connecting the selected points.)

The difference between (b) and (c) is as follows. In (b) the subsequence may be coded as a new sequence, and its location within the original sequence discarded, whereas in (c) the program should actually calculate the indices of the original sequence at which the desired subsequence occurs. That is, in the example above, should it output $3, 5, 6, 9$, the subsequence; or $1, 3, 4, 7$, the sequence of indices for the relevant subsequence?