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SUMS OF REGULAR SELFADJOINT OPERATORS

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ABSTRACT. This is an unedited transscript of the handwritten notes of my talk at the Master Class.

CONTENTS

1. PRELIMINARIES

1.1. Unbounded operators. An *unbounded operator* in a Hilbert space H is a linear map

$$T : \mathcal{D}(T) \subset H \longrightarrow H$$

defined on a (dense) subspace $\mathcal{D}(T)$ in H. Differential operators are inevitably unbounded when viewed as operators acting *in* a Hilbert space. An operator is called *closed* when its graph $\mathcal{G}(T)$ is a closed subpace of the product Hilbert space $H \times H$. For a closed operator $\mathcal{D}(T)$ itself becomes a Hilbert space when equipped with the *graph inner product* $\langle x, y\rangle_T := \langle x, y\rangle + \langle Tx, Ty\rangle$. Then T is a bounded operator from $\mathcal{D}(T)$ into H of norm ≤ 1. Thus the notion of an unbounded operator is more or less pointless when looking at maps between different spaces. T should instead be viewed as a self-map of the Hilbert space H. Only then it makes sense to talk about eigenvalues, spectral resolutions etc.

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Example 1.1. $D = i\frac{d}{dx}$ in $L^2[0,1]$.

$$\mathcal{D}(D_{\max/\min}) = H^1_{(0)}[0,1] \tag{1.1}$$

$$= \{f \in L^2[0,1] \mid f' \in L^2; \text{case min: } f(0) = f(1) = 0\} \tag{1.2}$$

$$\mathcal{D}(D_\lambda) = \{f \in \mathcal{D}(D_{\max}) \mid f(1) = \lambda \cdot f(0)\}, \quad \lambda \in S^1 \tag{1.3}$$

Eigenvalue equation: $Df = \mu \cdot f \Leftrightarrow f(x) = \text{const} \cdot e^{-i\mu x}$

Consequently: $\operatorname{spec} D_{\min} = \emptyset, \operatorname{spec} D_{\max} = \mathbb{C}, \operatorname{spec} D_{e^{2\pi i\alpha}} = 2\pi(\mathbb{Z} + \alpha)$

$D_{\min} \subset D_{\max} = D^*_{\min}$, thus $D_{\min}$ is *symmetric*.

$D^*_\lambda = D_\lambda$, thus D_λ is *self-adjoint*.

1.1.1. *Facts about a closed operator* T *in* H. 1. $H \oplus H = \mathcal{G}(T) \oplus J\mathcal{G}(T^*)$; $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

2. $I + T^*T$ is self-adjoint, invertible and $(I + T^*T)^{-1}$ is the adjoint of the natural inclusion $\iota_T : \mathcal{D}(T) \hookrightarrow H$, where $\mathcal{D}(T)$ is equipped with the natural graph Hilbert space structure.

3. If T is symmetric then T is self-adjoint if and only if $I \pm i \cdot T$ are onto. More precisely, one has the following orthogonal decompostion due to J. v. Neumann:

$$\mathcal{D}(T_{\max}) = \mathcal{D}(T_{\min}) \oplus \ker(T^* - i \cdot I) \oplus \ker(T^* + i \cdot I).$$

1.2. **Hilbert C^*-modules.** Let $\mathcal{A}$ be a C^*-algebra. Recall that a Hilbert $\mathcal{A}$-module E is an $\mathcal{A}$-right module with an $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle$ which is complete with respect to the norm $\|x\| := \|\langle x, x \rangle_{\mathcal{A}}^{\frac{1}{2}}\|$.

E feels and smells like a Hilbert space **BUT** there is almost never (except for $\mathcal{A} = \mathbb{C}$) a Projection Theorem. Try how far you get developing Hilbert space theory without a Projection Theorem.

Example 1.2. View $\mathcal{A}$ as a Hilbert module over itself and let $J \subset \mathcal{A}$ be an essential non-trivial ideal in $\mathcal{A}$. Then J is a proper Hilbert submodule of $\mathcal{A}$ with $J^\perp = \{0\}$.

1.2.1. *Semiregular (unbounded) operators.* A semiregular operator in E is a linear operator $T : \mathcal{D}(T) \longrightarrow E$ defined on a dense submodule $\mathcal{D}(T)$ and such that T^* (with respect to the $\mathcal{A}$-valued inner product) is densely defined. T is then automatically a closable module map.

However, the analogues of 1.1.1.1, 1.1.1.2, 1.1.1.3, do not come for free.

T is called *regular* if 1.1.1.1 or 1.1.1.2 (and hence both) holds. If T is symmetric 1.1.1.3 is equivalent to 1.1.1.1 or 1.1.1.2.

Problem: How to check regularity? Except, of course, solving the equation $(I + T^*T)x = y$ in a Hilbert module.

2. The Local-Global-Principle for regularity

2.1. **Localization.** Let $(\pi_\omega, H_\omega, \xi_\omega)$ be a cyclic representation of $\mathcal{A}$. We map E into the *interior tensor product* $E\widehat{\otimes}_{\mathcal{A}} H_\omega$ as follows:

$$\iota_\omega : E \longrightarrow E\widehat{\otimes}_{\mathcal{A}} H_\omega, x \mapsto x \otimes \xi_\omega$$

Recall the scalar product of the interior tensor product:

$$\langle \iota_\omega(x), \iota_\omega(y)\rangle_{E\widehat{\otimes}H_\omega} = \langle x\otimes\xi_\omega, y\otimes\xi_\omega\rangle_{E\widehat{\otimes}H_\omega} \tag{2.1}$$
$$= \langle \xi_\omega, \pi(\langle x, y\rangle_{\mathcal{A}})\xi_\omega\rangle_{H_\omega} \tag{2.2}$$
$$= \omega(\langle x, y\rangle_{\mathcal{A}}). \tag{2.3}$$

Thus $E\widehat{\otimes}_{\mathcal{A}}H_\omega$ is the completion of $E/\{x\in E \mid \langle x,x\rangle_\omega = 0\}$ with respect to the scalar product $\langle x,y\rangle_\omega := \omega(\langle x,y\rangle_{\mathcal{A}})$.

The semiregular operator T induces a closable operator T_0^ω as follows: $\mathcal{D}(T_0^\omega) := \mathcal{D}(T)\otimes_{\mathcal{A}}H_\omega = \mathcal{D}(T)\otimes_{\mathcal{A}}\xi_\omega$, $T_0^\omega(x\otimes\xi_\omega) := (Tx)\otimes\xi_\omega$. Its closure, T^ω, is the *localization of* T with respect to ω. It is trivial to see that $(T^*)^\omega \subset (T^\omega)^*$.

Equality is non-trivial and equivalent to regularity. This is the content of the Local-Global-Principle.

Theorem 2.1 (Local-Global-Principle; Pierrot, Kaad-Lesch)**.** *A closed semiregular operator* T *is regular if and only if for each (pure) state* ω *one has* $(T^*)^\omega = (T^\omega)^*$.

A densely defined symmetric operator T *is selfadjoint and regular if and only if for each (pure) state* ω *the operator* T^ω *is selfadjoint.*

The case of a general operator can easily be reduced to that of a symmetric operator by considering $\begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}$ instead.

2.1.1. *Example.* Let $D = i\frac{d}{dx}$ as before. Furthermore, let $X = [a,b]$, $E := L^2[0,1]\otimes C(X)$. This is the standard module $C(X)_{L^2[0,1]}$. It is convenient to view its elements as continuous functions $X \to L^2[0,1]$ with inner product

$$\langle f, g\rangle_{C(X)}(p) = \int_0^1 \overline{f(p,u)}\cdot g(p,u)du.$$

Let $\Lambda : [a,b] \to S^1$ be a Borel function, continuous on $(a,b]$ and such that the limit $\lim_{p\to a}\Lambda(p)$ does *not* exist. Put

$$\mathcal{D}(T_{\max/\min}) := \mathcal{D}(D_{\max/\min})\otimes_{C(X)} C(X), (Tf)(p) := D\big(f(p,\cdot)\big), \tag{2.4}$$
$$\mathcal{D}(T_\Lambda) := \big\{f\in\mathcal{D}(T_{\max}) \;\big|\; f(\cdot,1) = \Lambda\cdot f(\cdot,0)\big\}. \tag{2.5}$$

Then it is easy to see that T_Λ is semiregular as $\mathcal{D}(D_{\min})\otimes_{C(X)} C(X)$ is dense and contained in $\mathcal{D}(T_\Lambda)\cap\mathcal{D}(T_\Lambda^*)$.

Each $p\in[a,b]$ gives rise to a pure state on $C(X)$ and consequently to localizations T_Λ^p and $(T_\Lambda^*)^p \subset (T_\Lambda^p)^*$. If $f\in\mathcal{D}(T_\Lambda)$ then $f(p,\cdot)\in\mathcal{D}(T_{\Lambda,0}^p)\subset\mathcal{D}(D_{\Lambda(p)})$. Thus $T_\Lambda^p \subset D_{\Lambda(p)}$. Since evaluation at a point is continuous on the Sobolev space $H^1[0,1] = \mathcal{D}(D_{\max})$ we conclude that if there exists $f\in\mathcal{D}(T_\Lambda^p)$ with $f(p,0)\neq 0$ then $\Lambda(\cdot) = f(\cdot,1)/f(\cdot,0)$ is continuous in a neighborhood of p. Since Λ does not even have a limit at $p = a$ this implies that $T_\Lambda^a = D_{\min}$.

Fix two functions $\phi_0, \phi_1 \in C^\infty([0,1])$ such that $\operatorname{supp}(\phi_0)\subset[0,1/3)$, $\operatorname{supp}(\phi_1)\subset(2/3,1]$ and $\phi_j(j) = 1$. If $p\in(a,b]$ choose a cut-off function $\varphi\in C_c(a,b]$ with $\varphi(p) = 1$. Then $f(q) := \varphi(q)\cdot(\Lambda(q)\cdot\phi_1+\phi_0)$ lies in $\mathcal{D}(T_\Lambda)$ and $f(p,0)\neq 0$. This shows that $T^p_\Lambda \supset D_{\Lambda(p)}$ and hence with the proven converse inclusion $T^p_\Lambda = D_{\Lambda(p)}$ for $a < p \le b$.

We have $D_{\Lambda(p)} = (T^p_\Lambda)^* \supset (T^*_\Lambda)^p$. On the other hand the function f constructed before also lies in $\mathcal{D}(T^*_\Lambda)^p$, hence we have shown that for $a < p \le b$ we have $T^p_\Lambda = (T^*_\Lambda)^p = D_{\Lambda(p)}$.

We show that $(T^*_\Lambda)^a = D_{\min}$ as well. Since this is strictly smaller than $(T^a_\Lambda)^* = D^*_{\min} = D_{\max}$ this proves that T_Λ is not regular. To this end let $f \in \mathcal{D}(T^*_\Lambda)$. Let $\mathbf{1}$ be the constant function 1. Then

$$\begin{aligned}\langle \mathbf{1}, T^*_\Lambda f\rangle(p) &= \int_0^1 i f'(p,u)du \\ &= i\big(f(p,1) - f(p,0)\big) = i\big(\Lambda(p) - 1\big)\cdot f(p,0).\end{aligned}$$

If $f(a,0) \neq 0$ then Λ would have a limit as $p \to a$. By assumption this is not the case, hence we conclude that indeed $(T^*_\Lambda)^a = D_{\min}$.

Finally, let $f \in \mathcal{D}(T^*_\Lambda)$. By what we have proved so far we see that for each $p \in [a,b]$ we have $f(p,\cdot) \in \mathcal{D}(T^*_\Lambda)^p$ and thus $f(p,1) = \Lambda \cdot f(p,0)$ for *all* p since $f(a,0) = f(a,1) = 0$. But this means $f \in \mathcal{D}(T_\Lambda)$.

Altogether we have proved that T_Λ is selfadjoint but not regular.

2.2. **Application: Kato-Rellich-Wüst.** Let T be selfadjoint and regular and let V be a symmetric operator with $\mathcal{D}(V) \supset \mathcal{D}(T)$. Kato-Rellich type results give criteria for the sum $T + V$ being (essentially) selfadjoint if V is "small" compared to T. V is called relatively T–bounded if

$$\|Vx\| \le a \cdot \|Tx\| + b \cdot \|x\|$$

for $x \in \mathcal{D}(T)$. The constant a is called the relative bound. The constant b is unimportant.

The classical *Kato-Rellich Theorem* states that if $a < 1$ (and if E is a Hilbert space) that then $T + V$ is selfadjoint on $\mathcal{D}(T)$. The proof basically consists of a Neumann series argument which carries over verbatim to the Hilbert module setting. Wüst's extension of the classical Kato-Rellich Theorem states that if $a = 1$ then $T + V$ is essentially selfadjoint on $\mathcal{D}(T)$. The Hilbert space proof makes use of the Riesz representation Theorem and weak-*-compactness, therefore it does not carry over to the Hilbert C^*-module case. However, using the Local-Global Principle one easily infers the following

Theorem 2.2. *Let T be a selfadjoint and regular operator in the Hilbert $\mathcal{A}$-module E and let V be a symmetric operator in E with $\mathcal{D}(V) \supset \mathcal{D}(T)$ satisfying*

$$\langle Vx, Vx\rangle_{\mathcal{A}} \le \langle Tx, Tx\rangle_{\mathcal{A}} + b\langle x, x\rangle_{\mathcal{A}} \tag{2.6}$$

for all $x \in \mathcal{D}(T)$. Then $T + V$ is essentially selfadjoint on $\mathcal{D}(T)$ and its closure is regular.

Note that Eq. (2.6) is an inequality in the C^*-algebra $\mathcal{A}$.

Proof. For each state ω of $\mathcal{A}$ apply Kato-Rellich-Wüst to the localizations V^ω, T^ω. □

2.3. **Sketch of proof of the Local-Global-principle.** Suppose that T is selfadjoint but not regular. Then, say, $\operatorname{ran}(T+i) \subsetneq E$ is a proper closed submodule. For each state ω one easily checks $\operatorname{ran}(T^\omega + i) = \overline{\operatorname{ran}(T+i) \otimes \xi_\omega}$. Thus we need to find a (pure) state with $\operatorname{ran}(T+i) \otimes \xi_\omega$ not dense.

Theorem 2.3 (Kaad-Lesch; Pierrot). *Let* $L \subset E$ *be a proper closed submodule. Then for* $x_0 \in E \setminus L$ *there exists an (irreducible) cyclic representation* (H_ω, ξ_ω) *of* $\mathcal{A}$ *such that* $x_0 \otimes \xi_\omega \notin \overline{L \otimes \xi_\omega)}$.

In the talk I sketched two proofs of this Theorem, which implies the Local-Global-principle: the one given in the paper with Jens which is based on convexity arguments (Hahn-Banach). This proof is very elementary but does not show that the cyclic representation ω can be choosen to be irreducible.

The second proof I gave is my interpretation of Pierrot's argument. I say interpretation because Theorem 2.3 is not stated explicitly by Pierrot. The details of this proof will be published elsewhere as an erratum to the paper with Jens.

3. Sums of regular self-adjoint operators

We consider a pair of selfadjoint regular operators S, T in a Hilbert $\mathcal{A}$-module. One should think of S being $D_1 \otimes \mathbf{1}$ and T being $1 \otimes_\nabla D_2$. Under certain smallness assumptions on the commutator $[S, T]$ (there is also a domain problem here, of course) one should be able to conclude that the sum $\begin{pmatrix} 0 & S+iT \\ S-iT & 0 \end{pmatrix}$ is (essentially) selfadjoint and regular. This is motivated by the unbounded Kasparov product. Another more down to earth prototype is the Spectral Flow Theorem, which informally reads as follows: let $A(x)$ be a family of selfadjoint Fredholm operators parametrized by $x \in \mathbb{R}$. Then under certain conditions (there are many versions, so let us be vague here) the operator

$$\frac{d}{dx} + A(x) = \underbrace{A(x)}_{S} - i \cdot \underbrace{i\frac{d}{dx}}_{T}$$

is Fredholm and its index equals the Spectral Flow of the family $A(x)_x$.

We will investigate the regularity problem for $S+T$ under the following assumptions:

(1) There is a core $\mathcal{E} \subset \mathcal{D}(T)$ for T such that
(2) $(S-z)^{-1}(\mathcal{E}) \subset \mathcal{D}(S) \cap \mathcal{D}(T)$ for $z \in i\mathbb{R}^*, \mathbb{R}^* := \mathbb{R} \setminus \{0\}$,
(3) $[T, (S-z)^{-1}]_{|\mathcal{E}} =: X_{z,0}$ extends by continuity to a bounded adjointable map X_z in E.

As usual we let $(S \pm iT)_{\min}$ be the closure of $S \pm iT$ a priory defined on $\mathcal{D}(S) \cap \mathcal{D}(T)$ and $(S \pm iT)_{\max} = (S \mp iT)^*$. We abbreviate $D := S + iT$, $D_{\max/\min} = (S+iT)_{\max/\min}$.

One checks the following facts:

1. To check (3) it suffices to check boundedness, the adjointability comes for free here.
2. $(S-z)^{-1}$ preserves $\mathcal{D}(T), \mathcal{D}(S \pm iT)_{\max/\min}$

3. $X_z^* = -X_{\overline{z}}$
4. $\mathcal{E}$ can be replaced by $\mathcal{D}(T)$
5. Let P be a selfadjoint regular operator in a Hilbert C^*-module. Furthermore, let $(f_n) \subset C_b(\mathbb{R})$ be a sequence of bounded continuous functions such that $\sup_n \|f\|_\infty < \infty$ and $f_n \to f$ uniformly on compact subsets of $\mathbb{R}$. Then $f_n(P)$ converges *strongly* to $f(P)$.
6. $\mathcal{D}(S) \cap \mathcal{D}(T)$ is a core for S: to see this apply the previous item to see first that $-z(S-z)^{-1}$ converges strongly to the identity as $|z| \to \infty, z \in i\mathbb{R}$. Secondly, for $x \in \mathcal{D}(S)$ we have by (2) that $-z(S-z)^{-1}x \in \mathcal{D}(S) \cap \mathcal{D}(T)$. Furthermore, $-z(S-z)^{-1}x \to x$ and $S\big(-z(S-z)^{-1}x\big) = -z(S-z)^{-1}\big(Sx\big) \to Sx$, as $|z| \to \infty, z \in i\mathbb{R}$.
7. **Warning:** The same argument does **not quite** work to conclude that $\mathcal{D}(S) \cap \mathcal{D}(T)$ is a core for T. Namely, $T\big(-z(S-z)^{-1}x\big) = -zX_z x + (S-z)^{-1}\big(Tx\big)$. The second summand converges to Tx, but a priori we cannot say anything about the convergence of zX_z.

The last point is rather crucial and the following considerations somehow revolve around this problem.

Definition 3.1. *Let E be a Hilbert $\mathcal{A}$-module and let P be a regular operator in E. A submodule $\mathcal{E} \subset \mathcal{D}(P)$ is a* weak core *for P if the following holds: for each $x \in \mathcal{D}(P)$ there exists a sequence $(x_n) \subset \mathcal{E}$ such that $x_n \overset{\mathcal{A}}{\rightharpoonup} x$ and $\sup_n \|Px_n\| < \infty$. Here $x_n \overset{\mathcal{A}}{\rightharpoonup} x$ means that $\langle x_n, y\rangle \to \langle x, y\rangle$ for all $y \in \mathcal{A}$, generalizing weak convergence in a Hilbert space.*

One checks that if (x_n) satisfies the condition in the definition that then also $Px_n \overset{\mathcal{A}}{\rightharpoonup} Px$.

In a Hilbert space every weak core is automatically a core. The proof uses that in a Hilbert space a subspace is dense if its orthogonal complement is zero. Thus in a general Hilbert $\mathcal{A}$-module it is expected that the two notions of a core differ. However, at the moment I do not have an example for this claim.

Theorem 3.2 (Green's Formula)**.** *Under the general assumptions* (1)-(3) *from the beginning of this section assume that one of the following assumptions holds:*

(1) $\sup_{z \in i\mathbb{R}, |z| \geq 1} \|z \cdot X_z\| < \infty$.
(2) $\mathcal{D}(S) \cap \mathcal{D}(T)$ *is a weak core for* T.
(3) *For* $z \in i\mathbb{R}^*$ *we have* $(S-z)^{-1}\big(\mathcal{D}(D_{max})\big) \subset \mathcal{D}(S) \cap \mathcal{D}(T)$
(4) $\mathcal{D}(D_{max}) \cap \mathcal{D}(S) \subset \mathcal{D}(S) \cap \mathcal{D}(T)$.

Then the following Green's Formula holds for $x \in \mathcal{D}(D_{max}), y \in \mathcal{D}(D^t_{max})$:

$$\langle D_{max}x, y\rangle - \langle x, D^t_{max}y\rangle = \lim_{|z| \to \infty, z \in i\mathbb{R}} \langle -iz \cdot X_z x, y\rangle. \tag{3.1}$$

Note that this includes that the limit on the RHS exists for all such x, y.

The conditions (1)-(4) are not independent. In fact, (1)⇒(2)⇒(3)⇔(4).

Under the strongest assumption (1) it follows that the RHS of Eq. (3.1) *vanishes and hence that the operator*

$$\begin{pmatrix} 0 & S + iT \\ S - iT & 0 \end{pmatrix}$$

is essentially selfadjoint and regular on $\mathcal{D}(S) \cap \mathcal{D}(T)$.

Remark 3.3. This Theorem is slightly stronger than what is stated and proved in my paper with Jens Kaad in JFA.

The last claim that under (1) the RHS of Green's formula vanishes is easy to see: namely, first one notes from the very definition that the limit vanishes if $x, y \in \mathcal{D}(S) \cap \mathcal{D}(T)$. Since $\mathcal{D}(S) \cap \mathcal{D}(T)$ is dense in E it then follows from (1) by a standard $\varepsilon/3$ argument that the limit even exists and vanishes for *all* $x, y \in E$.

The essential selfadjointness of the last claim is now clear and the regularity is an immediate consequence of the Local-Global-Principle as all localizations are, by the same argument, essentially selfadjoint.

3.1. **An instructive example.** We will discuss an example where (3) and (4) are satisfied but not (1) or (2). Let $T = i\frac{d}{dx}$ acting in $L^2[0,1]$ with periodic boundary conditions. That is $\mathcal{D}(T) = \{f \in H^1[0,1] \mid f(1) = f(0)\}$. Fix a real number λ, $0 < |\lambda| < 1/2$ and a smooth function $\rho \in C^\infty((0,1))$ such that

$$\rho(x) = \begin{cases} \frac{\lambda}{x}, & 0 < x < 1/3, \\ \frac{\lambda}{1-x}, & 2/3 < x < 1, \end{cases} \tag{3.2}$$

and let $S = M_\rho$ be the multiplication operator by ρ.

The parameter λ is choosen deliberately since from the theory of Fuchs type differential operators, as can be found e.g. in the work of Cheeger or Brüning and Seeley, it is known that $\dim \mathcal{D}\big((S+iT)_{\max}\big)/\mathcal{D}\big((S+iT)_{\min}\big) = 2$. If $|\lambda| \geq 1/2$ then this dimension would be 0.

As before, let $D = S + iT$. Near $x = 0$ we have

$$(\rho(x) - z)^{-1} = \frac{x}{\lambda - xz},$$

(and similarly near $x = 1$), hence $(\rho(x) - z)^{-1}$ vanishes to first order at $x = 0$ and at $x = 1$. Furthermore, near $x = 0$

$$\frac{\rho'(x)}{(\rho(x) - z)^2} = \frac{-\lambda}{(\lambda - xz)^2}.$$

This shows that for $z \in i\mathbb{R}^*$ the operator $(S - z)^{-1}$ maps $H^1[0,1]$ continuously into $H^1_0[0,1] \subset \mathcal{D}(S) \cap \mathcal{D}(T)$. This shows that the pair S, T satisfies the general assumptions (1)-(3) from the beginning of this section.

The domain of $D_{\max}$ can be described explicitly: if $f \in \mathcal{D}(D_{\max})$, then

$$f(x) = c_0 \cdot x^{-\lambda} + f_1(x), \quad f_1(x) = O(x^{1/2}), \text{ as } x \to 0,$$

and $f_1 \in \mathcal{D}(D_{\min})$, and similarly as $x \to 1$. This and the proven mapping properties of S then imply that the pair S, T satisfies (3) and (4) of Theorem 3.2. Since in the Hilbert space setting every weak core is also a core, the pair neither satisfies (2) nor (1). Namely, $\mathcal{D}(S) \cap \mathcal{D}(T) \subset H^1_0[0,1]$ and this is not a (weak) core for T. In fact one sees that $|z| \cdot \|X_z\| \sim |z|$, as $z \to \infty$.

The boundary term in Green's formula can be computed explicitly. Instead of giving the general result, we give an example showing that the RHS of Green's formula is nontrivial and hence that $\begin{pmatrix} 0 & S+iT \\ S-iT & 0 \end{pmatrix}$ is not essentially selfadjoint.

Fix a cut-off function $\varphi \in C^\infty([0,1])$ which is constant 1 near 0 and which vanishes for $x \geq 1/3$. Put

$$f_1(x) := x^\lambda \cdot \varphi, \quad f_2(x) := x^{-\lambda} \cdot \varphi.$$

Then

$$(S+iT)_{\max} f_1 = -x^\lambda \cdot \varphi'(x), (S-iT)_{\max} f_2 = x^{-\lambda} \cdot \varphi'(x),$$

and thus

$$\begin{aligned} \langle f_1, (S-iT)_{\max} f_2 \rangle - \langle (S+iT)_{\max} f_1, f_2 \rangle &= \lim_{\varepsilon \to 0} \int_\varepsilon^1 2x^\lambda \varphi(x) \cdot x^{-\lambda} \varphi'(x) dx \\ &= 2 \int_0^1 \varphi \varphi' = -1. \end{aligned}$$

Just for fun let us compute the RHS of Green's formula directly:

$$\begin{aligned} \int_0^1 -izX_z f_1 f_2 &= \int_0^\infty \frac{\lambda \mu i}{(\lambda - i\mu x)^2} \varphi(x)^2 dx, z = \mu i \\ &= \int_0^\infty \frac{\lambda i}{(\lambda - i\mu)^2} \varphi(u/\mu)^2 du \\ &\stackrel{\mu \to \infty}{\longrightarrow} \varphi(0)^2 \int_0^\infty \frac{\lambda i}{(\lambda - i\mu)^2} du = -1, \end{aligned}$$

as it should be.

4. Dirac-Schrödinger operators – a variant of the Robbin-Salamon Theorem

TODO.

5. Literature

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