Descent and index pairings in weighted KK-theory

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*A G-***equivariant Kasparov module** (π, X, F) *(from A to B) consists of*

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The direct sum of G-equivariant Kasparov modules provides $KK^G(A, B)$ with the structure of an abelian group, which depends contravariantly on A and covariantly on B.

Descent in KK-theory

Theorem (Kasparov)

1. *There exists an explicit functorial group homomorphism*

$$\mathrm{Des} : KK^G(A, B) \to KK(C_r^*(G, A), C_r^*(G, B))$$

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2. *The relevant C^*-correspondence $C_r^*(G, X)$ is the completion of $C_c(G, X)$ with respect to the $C_r^*(G, B)$-valued inner product*

$$\langle \xi, \eta \rangle(k) := \sum_{g \in G} g^{-1}(\langle \xi(g), \eta(gk) \rangle)$$

The index pairing (even case)

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4. $$\mathrm{Ad}_W : K_0(\mathscr{K}(X)) \to K_0\big(\mathscr{K}(\ell^2(\mathbb{N}, B))\big) \cong K_0(B)$$

   is induced by any bounded adjointable isometry $W : X \to \ell^2(\mathbb{N}, B)$.

Main challenge

Challenge

Understand the composition of group homomorphisms

$$KK^G(\mathbb{C},\mathbb{C}) \xrightarrow{\text{Ind} \circ \text{Des}} \text{Hom}\big(K_*(C_r^*(G)), K_*(C_r^*(G))\big)$$

(show that a certain element $\gamma \in KK^G(\mathbb{C},\mathbb{C})$ *induces the identity homomorphism).*

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$$\mathrm{Des}^w : KK^{G,w}(\mathbb{C},\mathbb{C}) \to KK^w(C_c(G), C_r^*(G)) \qquad \text{and}$$

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3. *Obtaining a commutative diagram:*

$$\begin{array}{ccc} KK^G(\mathbb{C},\mathbb{C}) & \xrightarrow{\mathrm{Ind}\circ\mathrm{Des}} & \mathrm{Hom}\big(K_*(C_r^*(G)), K_*(C_r^*(G))\big) \\ \downarrow & & \| \\ KK^{G,w}(\mathbb{C},\mathbb{C}) & \xrightarrow{\mathrm{Ind}^w\circ\mathrm{Des}^w} & \mathrm{Hom}\big(K_*(C_r^*(G)), K_*(C_r^*(G))\big) \end{array}$$

G-weights

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*Let $\mathscr{A} \subseteq A$ be a dense G-invariant subalgebra. A G-**weight** on X is given by an unbounded positive, regular operator $D : \mathrm{Dom}(D) \to X$ such that*

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3. *The n-fold iterated commutator*

$$\Big[D, \ldots, \big[D, [D, a]\big] \ldots\Big] : \mathrm{Dom}(D^n) \to X$$

extends to a bounded adjointable operator $\delta^n(a)$ for all $a \in \mathscr{A}$, $n \in \mathbb{N}_0$.

Actions on domains

Definition

Let $n \in \mathbb{N}_0$*.*

1. *Define* $X_n := \mathrm{Dom}(D^n)$ *as Hilbert* C^**-module over* B *with inner product*

$$\langle \xi, \eta \rangle_n := \langle D^n \xi, D^n \eta \rangle + \langle \xi, \eta \rangle$$

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2. *Define* $A(D, n) \subseteq A$ *completion of* $\mathscr{A}$ *with respect to Banach algebra norm*

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Proposition

Left-actions of G *and* $A(D,n)$ *on* X *restricts to actions of* G *and* $A(D,n)$ *on* X_n.

Weighted equivariant Kasparov modules

Definition

A **weighted** G**-equivariant Kasparov module** (X, D, F) *(from* $\mathscr{A}$ *to* B*) of level* $n_0 \in \mathbb{N}_0$ *consists of*

such that

for all $m \geq n \geq n_0$, $a \in A(D, n)$ *and* $g \in G$.

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The **weighted** G-**equivariant** KK-**theory** *from* $\mathscr{A}$ *to* B *of* **level** $n_0 \in \mathbb{N}_0$ *is the quotient*

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Definition

The **weighted** G-**equivariant** KK-**theory** *from* $\mathscr{A}$ *to* B *is the direct limit*

$$KK^{G,w}(\mathscr{A}, B) := \lim_{n_0 \to \infty} KK^{G,w,n_0}(\mathscr{A}, B)$$

over the forgetful maps $\mathbb{E}^{G,w,n_0}(\mathscr{A}, B) \to \mathbb{E}^{G,w,m_0}(\mathscr{A}, B)$, $m_0 \geq n_0 \geq 0$.

Basic properties

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Proposition

For any G-invariant dense subalgebra $\mathscr{A} \subseteq A$, there exists a forgetful, functorial group homomorphism $\phi : KK^G(A, B) \to KK^{G,w}(\mathscr{A}, B)$ induced by $(X, F) \mapsto (X, 1, F)$.

Descent

Theorem (K.)

There exists a functorial group homomorphism

$$\mathrm{Des}^w : KK^{G,w}(\mathscr{A}, B) \to KK^w(C_c(G, \mathscr{A}), C_r^*(G, B))$$

compatible with descent in KK-theory via the forgetful homomorphism.

Index pairings

Theorem (K.)

There exists a functorial group homomorphism

$$\mathrm{Ind}^w : KK^w(\mathscr{A}, B) \to \mathrm{Hom}(K_*(A), K_*(B))$$

compatible with the index pairing in KK-theory via the forgetful homomorphism.

Compatibility with KK-theory

Theorem (K.)

The following diagram commutes:

$$\begin{array}{ccc}
KK^G(A,B) & \xrightarrow{\mathrm{Ind}\circ\mathrm{Des}} & \mathrm{Hom}(K_*(C_r^*(G,A)),K_*(C_r^*(G,B))) \\
\phi\big\downarrow & & \big\| \\
KK^{G,w}(\mathscr{A},B) & \xrightarrow{\mathrm{Ind}^w\circ\mathrm{Des}^w} & \mathrm{Hom}(K_*(C_r^*(G,A)),K_*(C_r^*(G,B)))
\end{array}$$

Applications

WE WILL SEE!