Quantum Differentiability of Essentially Bounded Functions on Euclidean Space

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August 25, 2016

Introducing Non-commutative Geometry

What is noncommutative geometry?

The study of noncommutative algebras which resemble algebras of functions on geometric spaces, using the methods and the language of geometry.

Non-commutative geometry

Non-commutative geometry in analysis is usually the study of algebras of operators on Hilbert space. Algebras of operators are considered the generalisation of algebras of functions.

Example

The algebra $L^\infty(\mathbb{R})$ is an algebra of bounded operators on $L^2(\mathbb{R})$. The full algebra $\mathcal{B}(L^2(\mathbb{R}))$ is the "noncommutative extension" of the study of functions on $\mathbb{R}$.

(Briefly) Quantised Calculus

Broadly speaking, calculus is the study of functions on $\mathbb{R}^d$.

Philosophy

Every locally integrable function f on $\mathbb{R}^d$ determines a (potentially unbounded) operator M_f on $L^2(\mathbb{R}^d)$.
In fact virtually every object of classical calculus arises in this way.
For example, differentiations ∂_j are operators on $L^2(\mathbb{R}^d)$ and

$$[\partial_j, M_f] = M_{\partial_j f}.$$

Quantised Calculus

Suppose we take this point of view seriously, that calculus is the study of operators on $L^2(\mathbb{R}^d)$.
This means that the class of "functions" is now expanded to $\mathcal{L}(L^2(\mathbb{R}^d))$.
$\mathcal{L}(L^2(\mathbb{R}^d))$ contains many new objects which have no analogue in usual real analysis.
(In fact we will typically expand our point of view to operators on $L^2(\mathbb{R}^d, \mathbb{C}^N)$)

Infinitesimals

Infinitesimals have been a part of mathematics since nearly the very beginning (since at least the 5th Century BC, from the Eleatic school in Greek Italy)

Definition (Infinitesimal, informal)

An infinitesimal ε is a quantity such that for all positive integers n,

$$0 \leq |\varepsilon| < \frac{1}{n}.$$

Expressed geometrically, this means that a line segment of length ε, when appended end to end any finite number of times, will never exceed 1 in length.

Obviously the only infinitesimal in $\mathbb{R}$ is zero.

Uses of nonzero infinitesimals

Mathematicians historically liked to pretend that $\mathbb{R}$ has nonzero infinitesimals. They found the concept extremely useful, such as in the following definitions:

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at x if for all infinitesimals ε, the difference $f(x+\varepsilon) - f(x)$ is infinitesimal.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be differentiable at a point x if for all infinitesimals ε, there is a constant c such that

$$f(x+\varepsilon) - f(x) - c\varepsilon$$

is infinitely smaller than ε. The difference $f(x+\varepsilon) - f(x)$ is usually called $df(x)$.

Infinitesimals as operators

Alain Connes noticed that $\mathcal{B}(H)$ contains elements not unlike infinitesimals:

Definition (Infinitesimal Operators)

An operator $T \in \mathcal{B}(H)$ is called *infinitesimal* if for any $\varepsilon > 0$ there is a finite dimensional subspace $E \subset H$ such that

$$\| T|_{E^\perp} \| \leq \varepsilon.$$

It is not hard to see that the set of infinitesimal operators is just the set of compact operators.

What properties should infinitesimals have?

Following the traditions of pre-rigorous analysis, we expect infinitesimals to have the following properties:

1. If $f : \mathbb{R} \to \mathbb{R}$ is a function, then f is continuous if df is infinitesimal
2. If f is smoother than g, then df is somehow "smaller" than dg
3. If x is a positive infinitesimal, then x^2 is "smaller" than x
4. If f is a differentiable function, then we can write $df = f'dx$ (provided that "sufficiently small" infinitesimals are ignored.

Provided that these vague properties are intepreted correctly, quantised calculus can give rigorous meaning to all of them. This talk focuses especially on 1 and 2.

To make sense of differentiability, we need to figure out what it means for one infinitesimal to be infinitely small compared to another.

Definition

The sequence of singular values $\{\mu(n,T)\}_{n=0}^{\infty}$ associated to a compact operator T is defined as

$$\mu(n,T) = \inf\{\|T-R\| \, : \, \mathrm{rank}(R) \leq n\}.$$

They are also the eigenvalues of $|T|$ in non-increasing order.

The rate of decay of $\{\mu(n,T)\}_{n=0}^{\infty}$ is a reasonable measure of the size of T.

Sizes of Infinitesimals

We can quantify the sizes an infinitesimal T by placing conditions on the rate of decay of $\{\mu_k(T)\}_{k=0}^{\infty}$.

- The smallest infinitesimals have $\{\mu_k(T)\}_{k=0}^{\infty}$ of finite support. Then T is of finite rank.
- We say that $T \in \mathcal{L}^p$ if $\{\mu_k(T)\}_{k=0}^{\infty} \in \ell^p$.
- We say that $T \in \mathcal{L}^{p,\infty}$ if $\mu_k(T) = \mathcal{O}(k^{-1/p})$.
- We say that $T \in \mathcal{L}^{p,q}$ if $\{k^{1/p-1/q}\mu_k(T)\}_{k=0}^{\infty} \in \ell^q$.
- We say that $T \in \mathcal{M}_{1,\infty}$ if $\{\frac{1}{\log(k+1)}\sum_{n=0}^{k}\mu_k(T)\}_{k=0}^{\infty} \in \ell^{\infty}$.

These last four conditions in fact correspond to ideals of $\mathcal{B}(H)$.

Sizes of Infinitesimals

Theorem

Let T be a compact operator in $\mathcal{B}(H)$. Then for every $k \geq 0$,

$$\mu_k(T^2) \leq \|T\|\mu_k(T).$$

Hence, if T is an infinitesimal, then T^2 is a smaller infinitesimal.

Notations

H denotes any complex separable Hilbert space.

- $\mathcal{B}(H)$ is the algebra of bounded operators on H.
- $\mathcal{L}_{p,q}(H)$ is the Schatten-Lorentz space of operators T with $\{\mu(n,T)\}_{n=0}^{\infty} \in \ell^{p,q}$.
- We work over $\mathbb{R}^d$, and always $d > 1$.
- $L^\infty(\mathbb{R}^d)$ acts on $L^2(\mathbb{R}^d)$ by pointwise multiplication as M_f.
- $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of functions on $\mathbb{R}^d$.
- We denote $-i\partial_j$ as D_j.

Riesz Transforms

For $j = 1, \ldots, d$, define

$$R_j = \frac{\partial_j}{\sqrt{\partial_1^2 + \partial_2^2 + \cdots + \partial_d^2}}.$$

R_j is called the jth Riesz transform. By functional calculus R_j is bounded on L^2.

Quantised Differentials

In noncommutative geometry, we have a new object that is not present in classical analysis called a quantised derivative or quantised differential.

Definition

Let $f \in L^\infty(\mathbb{R})$. M_f is the operator on $L^2(\mathbb{R})$ of pointwise multiplication: $M_f g(x) = f(x)g(x)$ for almost all $x \in \mathbb{R}$. F denotes the Hilbert transform:

$$Fg(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{sgn}(\xi) e^{ix\xi} \widehat{g}(\xi)\, d\xi.$$

(Defined at least initially for g a smooth function of compact support, then extended by continuity to $g \in L^2(\mathbb{R})$). Then we define:

$$đ f := [F, M_f].$$

What is a quantised differential?

$đf$ is supposed to represent the "infinitesimal variation", like df in classical analysis.

Warning:

$đf$ is not a one-form, or a derivative. It is an infinitesimal deviation. It should be thought of as $f(x+h) - f(x)$ for infinitesimal epsilon.

It is very hard to motivate the definition of $đf$. Instead, we will show that is satisfies a number of "headline properties" of a classical differential.

A dictionary

Classical Analysis	Non-commutative Analysis
Function	Operator
Range	Spectrum
Infinitesimal	Compact Operator
Differential	Quantised differential

Rediscovering Classical Definitions

Let $f:\mathbb{R}\to\mathbb{R}$ be Borel. f is continuous at $x\in\mathbb{R}$ if and only if $f(x+T)-f(x)$ is compact for all compact self-adjoint operators T.

Lipschitz continuity and $\mathcal{L}_p$ spaces

If $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz and ε is an informal infinitesimal, one should think that $f(x+\varepsilon)-f(x)$ is an infinitesimal of the same size. The following can be viewed as a rigorous justification:

Theorem (Potapov and Sukochev, 2001)

Let $p\in(1,\infty)$. A function $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz if and only if for all self adjoint $T\in\mathcal{L}_p$ and all x we have

$$\|f(x+T)-f(x)\|_{\mathcal{L}_p}\leq c_{\mathrm{abs}}\|f'\|_\infty\|T\|_{\mathcal{L}_p}.$$

Another approach to differentiability

Recall the classical differential df of a function f, defined as an infinitesimal variation in f.

Connes defines the following "quantised differential"

Definition (Quantised Differential on $\mathbb{R}$)

Let $f \in L^\infty(\mathbb{R})$.

$$đf = i[\mathrm{sgn}\left(-i\frac{d}{dx}\right), M_f].$$

The origin of the definition is in noncommutative geometry and is beyond the scope of this talk.

Quantised differentials

Let us just take for granted the definition of $\bar{d}f$, and try to work with it.

Conjecture

$\bar{d}f$ should be infinitesimal (i.e., compact) if f is continuous.

Conjecture

The size of $\bar{d}f$ (i.e., the rate of decay of the singular values) should somehow decrease with the smoothness of f.

Revision of Classical Fourier Analysis and Notation

We define $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\}$. Let $z : \mathbb{T} \to \mathbb{T}$ be the identity function. Denote the normalised Haar (or arc length) measure on $\mathbb{T}$ by $\boldsymbol{m}$.

For $f \in L^1(\mathbb{T}, \boldsymbol{m})$, define for $n \in \mathbb{Z}$,

$$\widehat{f}(n) := \int_{\mathbb{T}} z^{-n} f \, d\boldsymbol{m}.$$

Recall that any $f \in L^2(\mathbb{T}, \boldsymbol{m})$ can be written as

$$f = \sum_{n \in \mathbb{Z}} \widehat{f}(n) z^n.$$

The sum converges in the L^2 sense. This effects an isometric isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.

Revision of Classical Fourier Analysis and Notation

The closed linear span of $\{z^n\}_{n=0}^{\infty}$ in L^2 is denoted $H^2(\mathbb{T})$, and the orthogonal complement is denoted $H^2_-(\mathbb{T})$.
We define the space of polynomials $\mathcal{P}(\mathbb{T})$ to be the finite linear span of $\{z^n\}_{n\in\mathbb{Z}}$. $P_A(\mathbb{T}) = \text{span}\{z^n\}_{n\geq 0}$.

Differentials on $\mathbb{T}$

Defining quantised differentials for functions on $\mathbb{T}$ is exactly like for functions on $\mathbb{R}$.

Definition

The Hilbert transform, for $g \in L^2(\mathbb{T})$, is defined to be

$$Fg := \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\widehat{g}(n)z^n.$$

We define the quantised differential:

$$đf := [F, M_f].$$

Differentials on $\mathbb{T}$

Hence for $f \in L^2(\mathbb{T})$,

$$Ff = \varphi * f$$

where

$$\varphi = \sum_{n \in \mathbb{Z}} \mathrm{sgn}(n) z^n = \frac{1}{1-z} - \frac{z^{-1}}{1-z^{-1}} = \frac{2}{1-z}.$$

Thus,

$$\begin{aligned}(\bar{\partial} f)g &= ([F,f]g)(t) \\ &= 2 \lim_{\varepsilon \to 0} \int_{|\tau - t| > \varepsilon} \frac{f(t) - f(\tau)}{t - \tau} g(\tau)\, d\boldsymbol{m}(\tau).\end{aligned}$$

Finite rank differentials

Let $f:\mathbb{T}\to\mathbb{C}$. The strictest condition we can put on the smoothness of f is that f is a rational function. The strictest condition we can put on the size of $\bar{d}f$ is that $\bar{d}f$ is finite rank. These two conditions are equivalent.

Theorem (Kronecker)

If $f:\mathbb{T}\to\mathbb{C}$, then $\bar{d}f$ is finite rank if and only if f is a rational function.

Bounded differentials

Let $f : \mathbb{T} \to \mathbb{C}$. The weakest condition that we can place on $\bar{\partial} f$ is that $\bar{\partial} f$ is bounded.

Definition

Let $f : \mathbb{T} \to \mathbb{C}$ be measureable. We say that f is of *bounded mean oscillation* if for an arc $I \subseteq \mathbb{T}$, define

$$f_I = \frac{1}{\boldsymbol{m}(I)} \int_I f \, d\boldsymbol{m}$$

and

$$\sup_I \frac{1}{\boldsymbol{m}(I)} \int_I |f - f_I| \, d\boldsymbol{m} < \infty$$

where the supremum runs over all arcs I. The set of functions with bounded mean oscillation is denoted $\mathrm{BMO}(\mathbb{T})$.

Bounded differentials

Theorem (Nehari)

Let $f : \mathbb{T} \to \mathbb{C}$. Then $\bar{d}f$ is bounded if and only if $f \in \mathrm{BMO}(\mathbb{T})$.

Compact differentials

We define the space $\mathrm{VMO}(\mathbb{T})$:

Definition

We say that $f \in \mathrm{VMO}(\mathbb{T})$ if $f \in \mathrm{BMO}(\mathbb{T})$ and

$$\lim_{\boldsymbol{m}(I)\to 0} \frac{1}{\boldsymbol{m}(I)} \int_I |f - f_I| \, d\boldsymbol{m} = 0.$$

Theorem

If $f : \mathbb{T} \to \mathbb{C}$, then $\bar{d}f$ is compact if and only if $f \in \mathrm{VMO}(\mathbb{T})$.

Can we do better?

We seek a more precise characterisation of the relationship between the smoothness of f and the size of $\bar{d}f$. To this end, we define the *Besov classes* B^s_{pq}.

(Aside) Besov Spaces

Let $f \in L^\infty(\mathbb{R}^d)$.

Central Dogma of Harmonic Analysis

Smoothness of f corresponds to the rate of decay of $\hat{f}$ (and vice versa).

The Besov space $B^s_{p,q}(\mathbb{R}^d)$ consists of functions f such that f can be written as

$$f = \sum_{n\in\mathbb{Z}} f_n$$

converging in L^p such that $\mathrm{supp}(\hat{f}_n) \subseteq B(0,2^{n+1}) \setminus B(0,2^{n-1})$ and

$$\sum_{n\in\mathbb{Z}} 2^{|n|sq} \|f_n\|_p^q < \infty.$$

Results of Peller (and others)

These conjectures have amazingly nice answers:

Theorem

If $f \in C_0(\mathbb{R}) + \mathbb{C}$ then $\bar{d}f$ is compact.

Theorem

For $p \in (0,\infty)$, $\bar{d}f \in \mathcal{L}_p$ if and only if $f \in B^{1/p}_{p,p}(\mathbb{R})$.

There are similar (but less easily stated) theorems giving necessary and sufficient theorems for $\bar{d}f \in \mathcal{L}_{p,q}$ for $p \in (0,\infty)$ and $q \in (0,\infty]$.

Classical Limits

How can we justify the name "quantised differential"?
Let $f \in C_0(\mathbb{R}) + \mathbb{C}$. Then $|\bar{d}f|$ has a sequence of eigenvalues $\lambda_0 \geq \lambda_1 \geq \cdots \geq 0$, with $\lim_{n\to\infty} \lambda_n = 0$.
Quantum mechanically, the numbers $\{\lambda_n\}_{n=0}^{\infty}$ correspond to observable values of $|\bar{d}f|$. So their asymptotics must correspond to a classical limit (somehow)... Let ω be a dilation invariant extended limit on $\ell^\infty(\mathbb{N})$. Then,

$$\omega_{n\to\infty}\left(\frac{\lambda_0+\lambda_1+\cdots+\lambda_n}{\log(2+n)}\right) = c_{\mathrm{abs}}\int_{\mathbb{R}} |f'(x)|\,dx.$$

So $\bar{d}f$ corresponds to f' in the classical limit (in some sense).

Historical introduction: the normal trace and singular traces

Normal trace

Let $B(H)$ be an algebra of all bounded linear operators on a separable Hilbert space H.

Canonical trace:

$$\mathrm{Tr}(A) = \sum_{k=0}^{\infty} \lambda(k, A),$$

the sequence $\{\lambda(k, A)\}_{k=0}^{\infty}$ is an eigenvalue sequence of a compact operator A.

The normal trace is essentially unique.

Question: Are there any other traces besides the canonical?

The Dixmier's construction (1966)

Let $A \in B(H)$ be a positive compact operator s.t.

$$\sup_{n>0} \frac{1}{\log(1+n)} \sum_{k=0}^{n-1} \lambda(k,A) < \infty.$$

Let

$$t(A) := lim \frac{1}{\log(1+n)} \sum_{k=0}^{n-1} \lambda(k,A), \quad 0 \leq A.$$

Two problems: (i) convergence, (ii) additivity.

Setting *lim* to be a linear form invariant under the group of affine transformations $t \mapsto at + b$ on $\mathbb{R}$ solves both of these problems.

Non-normal traces

If ω is a dilation invariant singular state on the space l_∞ of bounded sequences, (that is, $\omega(x_0, x_1, \dots) = \omega(x_0, x_0, x_1, x_1, \dots)$) the functional

$$\mathrm{Tr}_\omega(A) := \omega\left(\frac{1}{\log(1+n)}\sum_{k=0}^{n-1}\lambda(k,A),\right), \quad 0 \le A,$$

is positively homogeneous and additive.

1. Tr_ω is a trace;
2. Tr_ω is non-trivial: $\mathrm{Tr}_\omega(\mathrm{diag}\left\{\frac{1}{n+1}\right\}) = 1$;
3. $\mathrm{Tr}_\omega(A) = 0$ if A is finite rank operator.

Hence, Tr_ω is a non-trivial, non-normal trace.

Dixmier traces and applications

Extended limits

To introduce the revised construction of Dixmier traces we require some preparations.

Definition

A linear functional ω on $L_\infty = L_\infty(0,\infty)$ is called an extended limit if

(i) $\omega(x) \geq 0$, whenever $0 \leq x \in L_\infty$;

(ii) $\omega(\chi_{(0,\infty)}) = 1$;

(iii) $\omega(x) = \lim_{t\to\infty} x(t)$ (if the limit exists).

Conditions (i)-(iii) are equivalent to the fact that ω is a Hahn-Banach extension of the usual limit functional.

Such functionals are singular, that is they vanish on compactly supported functions.

Invariant extended limits

Definition

An extended limit ω on L_∞ is called

- translation invariant if $\omega(T_h x) = \omega(x)$ for every $h > 0$;
- dilation invariant if $\omega(\sigma_\beta x) = \omega(x)$ for every $\beta > 0$,
- exponentiation invariant if $\omega(P^a x) = \omega(x)$ for every $a > 0$,

where

$$(T_h x)(t) := x(t+h), \quad (\sigma_\beta x)(t) := x(t/\beta), \quad (P^a x)(t) = x(t^a).$$

Later on another type of invariant extended limits will be introduced.

Singular value function

For every $A \in B(H)$ a generalized singular value function is defined by the formula

$$\mu(t, A) = \inf\{\|Ap\|_\infty : p \text{ is a projection in } B(H) \text{ with } \tau(1-p) \leq t\}.$$

If A is compact, then $\mu(k-1, A)$ is the k-th largest eigenvalue of an operator $|A| = (A^*A)^{1/2}$, $k \in \mathbb{N}$ and

$$\mu(t, A) = \sum_{n=0}^{\infty} \mu(n, A)\chi_{[n,n+1)}(t), \ \ t > 0.$$

Dixmier traces

Definition

The classical Dixmier-Macaev ideal:

$$\mathcal{M}_{1,\infty} := \left\{ A \text{ is compact} : \sup_{t>0} \frac{1}{\log(1+t)} \int_0^t \mu(s,A)\, ds < \infty \right\}.$$

For an arbitrary dilation invariant extended limit ω the functional

$$\mathrm{Tr}_\omega(A) := \omega\left(\frac{1}{\log(1+t)} \int_0^t \mu(s,A)\, ds \right), \quad 0 \leq A \in \mathcal{M}_{1,\infty},$$

extends to a non-normal trace (a Dixmier trace) on $\mathcal{M}_{1,\infty}$[1].

[1] A.Carey, F.Sukochev, Dixmier traces and some applications to noncommutative geometry, *Uspekhi Mat. Nauk*, 2006.

Dixmier trace is singular

If $0 \leq A \in \mathcal{L}_1$, then it follows from the definition of Dixmier trace that

$$\mathrm{Tr}_\omega(A) = \omega(\frac{O(1)}{\log(2+n)}) = 0.$$

In particular, Dixmier trace vanishes on every finite rank operator.

One can now see that Tr_ω is no normal. Indeed,

$$\mathrm{diag}(\{\frac{1}{k+1}\}_{0\leq k\leq n}) \uparrow \mathrm{diag}(\{\frac{1}{k+1}\}_{0\leq k})$$

as $n \to \infty$. However,

$$0 = \mathrm{Tr}_\omega(A)(\mathrm{diag}(\{\frac{1}{k+1}\}_{0\leq k\leq n})) \not\to \mathrm{Tr}_\omega(A)(\mathrm{diag}(\{\frac{1}{k+1}\}_{0\leq k})) = 1$$

Non-commutative integral

Let $(\mathcal{A}, H, D)$ be a spectral triple. Here, $\mathcal{A}$ is a $*$-algebra of bounded operators on the Hilbert space H and $D : \mathrm{Dom}(D) \to H$ is an unbounded self-adjoint operator, such that D^2 is a positive operator with compact resolvent. Assume that $(1 + D^2)^{-d/2} \in \mathcal{M}_{1,\infty}$ for some $d \in \mathbb{R}$. Hence, $a \cdot (1 + D^2)^{-d/2} \in \mathcal{M}_{1,\infty}$ for every $a \in \mathcal{A}$.

If all Dixmier traces coincide on $a \cdot (1 + D^2)^{-d/2}$, then one defines

$$\fint a := \mathrm{Tr}_\omega(a \cdot (1 + D^2)^{-d/2}).$$

Application: Connes Trace Theorem

Let $\Delta = \sum_{n=1}^{d} \frac{\partial^2}{\partial x_n^2}$ denotes the Laplace operator on $\mathbb{R}^d$.

If $f \in C_c^\infty(\mathbb{R}^d)$, then $M_f(1-\Delta)^{-d/2}$ is a classical pseudo-differential operator of order $-d$.

Hence[2], the operator $M_f(1-\Delta)^{-d/2}$ belongs to $\mathcal{M}_{1,\infty}$ (the domain of Dixmier traces) and

$$⨍ f = \mathrm{Tr}_\omega(M_f(1-\Delta)^{-d/2}) = \frac{\mathrm{Vol}\,\mathbb{S}^{d-1}}{d(2\pi)^d} \cdot \int_{\mathbb{R}^d} f(u)du,$$

for every Dixmier trace Tr_ω.

[2] A. Connes, The action functional in noncommutative geometry, *Comm. Math. Phys.*, 1988.

Generalisation of Connes Trace Theorem

The extension of CTT can be done in few directions:
- expand the class of functions f;
- extend the class of traces beyond Dixmier traces.

Theorem ([3],[4])

If $f \in L_2(\mathbb{R}^d)$ is compactly supported, then $M_f(1-\Delta)^{-d/2} \in \mathcal{M}_{1,\infty}$ and

$$⨍ f = \tau(M_f(1-\Delta)^{-d/2}) = \tau(\mathrm{diag}\{1/n\})\frac{\mathrm{Vol}\,\mathbb{S}^{d-1}}{d(2\pi)^d} \cdot \int_{\mathbb{R}^d} f(u)du,$$

for every trace (that is, unitarily invariant linear functional) τ on $\mathcal{M}_{1,\infty}$.

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[3] N.Kalton, S.Lord, D.Potapov, F.Sukochev, Traces of compact operators and the noncommutative residue, Adv. Math., 2013

[4] S.Lord, F.Sukochev, D.Zanin, *Singular Traces: Theory and Applications*, 2012

Generalisation of Connes Trace Theorem 2

Let X be a d-dimensional closed Riemannian manifold with metric g. Let Δ_g denotes the Laplace-Beltrami operator on $C^\infty(X)$. (Laplace-Beltrami operator is an extension of the notion of Laplace operator to manifolds. It is defined as the divergence of the gradient, as in the classical case.)

Theorem ([5],[6])

If $f \in L_2(X)$, then $M_f(1-\Delta_g)^{-d/2} \in \mathcal{M}_{1,\infty}$ and

$$⨍ f = \tau(M_f(1-\Delta_g)^{-d/2}) = \tau(\mathrm{diag}\{1/n\})\frac{\mathrm{Vol}\,\mathbb{S}^{d-1}}{d(2\pi)^d}\cdot\int_X f(u)du,$$

for every trace τ on $\mathcal{M}_{1,\infty}$.

[5] S.Lord, D.Potapov, F.Sukochev, Measures from Dixmier traces and zeta functions, JFA, 2010

[6] S.Lord, F.Sukochev, D.Zanin, *Singular Traces: Theory and Applications*, 2012.

Generalisation of Connes Trace Theorem 3

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be a torus. Let $\Delta = \frac{1}{4\pi}(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2})$ denotes the Laplace operator on $\mathbb{T}^2$.

Theorem ([7],[8])

If $f \in L_\infty(\mathbb{T}^2)$, then $M_f(1-\Delta)^{-1} \in \mathcal{M}_{1,\infty}$ and

$$⨍ f = \tau(M_f(1-\Delta)^{-1}) = c\int_{\mathbb{T}^2} f(u)du,$$

for every trace τ on $\mathcal{M}_{1,\infty}$, where $c > 0$ does not depend on τ.

[7] S.Lord, D.Potapov, F.Sukochev, Measures from Dixmier traces and zeta functions, JFA, 2010

[8] S.Lord, F.Sukochev, D.Zanin, *Singular Traces: Theory and Applications*, 2012.

Application: Banach space geometry

G. Pisier and Q. Xu[9] raised the question on the differentiability of the norms of non-commutative L_p-spaces associated with an arbitrary von Neumann algebra $\mathcal{M}$. They asked whether the L_p-norm of a noncommutative L_p-space has the same differentiability properties as the norm of a classical (commutative) L_p-space.

When the algebra $\mathcal{M}$ is of type I this question has been fully resolved[10], but the general case required new ideas.

[9] G. Pisier and Q. Xu, Non-commutative Lp-spaces, *Handbook of the geometry of Banach spaces*, 2003.

[10] D. Potapov and F. Sukochev, Fréchet differentiability of S^p norms, *Adv. Math.*, 2014.

Application: Banach space geometry 2

Theorem ([11])

Let $H \in \mathcal{L}_p(\mathcal{M})$. The function $H \mapsto \|H\|_p^p$ is

(i) *infinitely many times Fréchet differentiable, if $p \in 2\mathbb{Z}$;*

(ii) *$(p-1)$-times Fréchet differentiable, if $p \in 2\mathbb{Z}+1$;*

(iii) *$[p]$-times Fréchet differentiable, $p \notin \mathbb{Z}$.*

Here, Fréchet differentials are defined via singular traces on weak non-commutative L_1-spaces associated with semifinite (non-finite) von Neumann algebras.

[11] D. Potapov, F. Sukochev, A. Tomskova, D. Zanin, Fréchet differentiability of the norm of L_p-spaces, submitted manuscript.

Higher Dimensions 1

What is the appropriate analogue of $\bar{\partial} f$ when f is a function on $\mathbb{R}^d$?

Let $N = 2^{\lfloor d/2 \rfloor}$. $\{\gamma_j\}_{j=1}^{d}$ are $N \times N$ self-adjoint matrices satisfying $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{j,k}$, where δ is the Kronecker delta. The precise choice of matrices satisfying this relation is unimportant so we assume that a choice is fixed. Using this choice of gamma matrices, we can define the d-dimensional Dirac operator.

Higher Dimensions 2

Let $\mathcal{D}$ be the operator on $L^2(\mathbb{R}^d, \mathbb{C}^N)$ defined as

$$\mathcal{D} = \sum_{j=1}^{d} \gamma_j \otimes (-i\partial_j).$$

This is a linear operator on the Hilbert space $\mathbb{C}^N \otimes L^2(\mathbb{R}^d)$ initially defined with dense domain $\mathbb{C}^N \otimes \mathcal{S}(\mathbb{R}^d)$, where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of functions on $\mathbb{R}^d$. It is easily seen that $\mathcal{D}$ is symmetric on this domain. Taking the closure we obtain a self-adjoint operator which we also denote $\mathcal{D}$.

Higher Dimensions 3

We then define the sign of $\mathcal{D}$ as $sgn(D) = D|D|^{-1}$, $D = \sum_k \gamma_k \otimes D_k, |D| = 1 \otimes (\sum_k D_k^2)^{1/2}$. Thus,

$$\operatorname{sgn}(\mathcal{D}) := \sum_{j=1}^{d} \gamma_j \otimes \frac{D_j}{\sqrt{D_1^2 + D_2^2 + \cdots + D_d^2}}.$$

This is defined through the Borel functional calculus. Specifically, the operator $D_j/\sqrt{D_1^2 + D_2^2 + \cdots + D_d^2}$ is the result of applying the function $x \mapsto x_j/\|x\|$ to $\mathcal{D}$. Consequently $\operatorname{sgn}(\mathcal{D})$ extends to a bounded operator on $\mathbb{C}^N \otimes L^2(\mathbb{R}^d)$. Suppose $f \in L^\infty(\mathbb{R}^d)$. The operator $1 \otimes M_f$ is a bounded linear operator on $\mathbb{C}^N \otimes L^2(\mathbb{R}^d)$, where 1 here is considered as the identity operator on $\mathbb{C}^N$. Thus we can define,

$$đf := i[\operatorname{sgn}(\mathcal{D}), 1 \otimes M_f].$$

Smoothness in higher dimensions 1

What is the relationship between smoothness of $f \in L^\infty(\mathbb{R}^d)$ and the size of $\bar{d}f := i[\mathrm{sgn}(\mathcal{D}), 1 \otimes M_f]$?

In one dimension, necessary and sufficient conditions on $f \in L^\infty(\mathbb{R})$ such that $[\mathrm{sgn}(-id/dx), M_f] \in \mathcal{L}_{p,q}$ where $p, q \in [0, \infty]$ are provided by Peller.

Janson and Wolfe (1982), Connes, Sullivan and Teleman (1994) have studied necessary and sufficient conditions for $\bar{d}f \in \mathcal{L}_{p,q}$ with $p, q \in [0, \infty]$ in the higher dimensional case, $d > 1$.

Smoothness in higher dimensions 2

Theorem (Janson and Wolff)

if $0 < p \leq d$, we have that $\bar{d}f \in \mathcal{L}_p$ if and only if f is constant. For $p > d$, we have $\bar{d}f \in \mathcal{L}_p$ if and only if $f \in B^{d/p}_{p,p}(\mathbb{R}^d)$.

The case of $p \neq q$ with $p, q \in [1, \infty)$ was answered by Rochberg and Semmes in 1989. Necessary and sufficient conditions on $f \in L^\infty(\mathbb{R}^d)$ are given so that $\bar{d}f \in \mathcal{L}_{p,q}$ with $p \in [1, \infty)$ and $q \in [1, \infty)$. These conditions are given in terms of the mean oscillation of f, and it is not obvious whether an equivalent condition could be given in terms of more familiar function spaces.

Smoothness in higher dimensions 3

It is of interest in Connes' quantised calculus to determine conditions on f such that $\bar{d}f \in \mathcal{L}_{d,\infty}(\mathbb{C}^N \otimes L^2(\mathbb{R}^d))$.
The asymptotic behaviour of the singular values of the quantised derivative denote the dimension of the infinitesimal in the quantised calculus. That the sequence of singular values belongs to the weak space $\mathcal{L}_{d,\infty}$ when the dimension of the Euclidean space is d indicates analogous behaviour between quantum derivatives and differential forms. Specifically, a product of d derivatives lies in the space $\mathcal{L}_{1,\infty}$ which is the only weak space admitting a non-trivial trace that acts as the integral.

Smoothness in higher dimensions 4

1. What conditions on f ensure that $\bar{d}f \in \mathcal{L}_{d,\infty}$?
2. Is there some "classical limit" relating $\bar{d}f$ to ∇f?

We can now answer these questions!

Commutators with order zero pseudodifferential operators

There is an interesting theorem in the Appendix of a 1994 paper of Connes, Sullivan and Teleman:

Theorem (Connes, Sullivan and Teleman)

Let T be an order zero pseudodifferential operator that is translation and dilation invariant, and let f be a function on $\mathbb{R}^d$, $d > 1$. Then the commutator $[T, M_f]$ is in $\mathcal{L}_{d,\infty}$ if and only if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$.

They provided a sketch proof. We reprove this result using new methods.

Our results 1

When we write $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$ we implicitly assume that the essentially bounded function f has odd weak partial derivatives and that the Bochner norm of ∇f in $L^d(\mathbb{R}^d, \mathbb{C}^d)$ given by

$$\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} = \left(\int_{\mathbb{R}^d} \|\nabla f(x)\|_d^d dx\right)^{1/d} = \left(\int_{\mathbb{R}^d} \sum_{j=1}^d |D_j f(x)|^d dx\right)^{1/d}$$

is finite.

Our results 2

Let $f \in L^\infty(\mathbb{R}^d)$. Then we have:

Theorem

$\bar{d}f \in \mathcal{L}_{d,\infty}$ if and only if $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$, and

$$\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \lesssim \|\bar{d}f\|_{d,\infty} \lesssim \|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.$$

Theorem

Let $|\bar{d}f|^d$ have eigenvalues $\lambda_0 \geq \lambda_1 \geq \cdots \geq 0$. Then for a dilation invariant extended limit ω,

$$\omega_{n\to\infty}\left(\frac{\lambda_0 + \cdots + \lambda_d}{\log(2+n)}\right) = c_d \|\nabla f\|^d_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.$$

where $c_d > 0$ depends on d.

How are these results obtained?

Our proofs use the techniques:

1. Double Operator Integrals
2. Pseudodifferential Operators
3. Singular traces

The plan of proof

The proof is done in three steps:

1. First we prove that $\|đf\|_{d,\infty} \leq K_d\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}$
2. Then we obtain the formula for $\varphi(|đf|^d)$
3. From the trace formula we finish the proof by obtaining $k_d\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \leq \|đf\|_{d,\infty}$

The first step

Now we show how to prove that:

$$\|\bar{d}f\|_{d,\infty} \leq K_d \|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.$$

First under the assumption that $f \in \mathcal{S}(\mathbb{R}^d)$, then extended to $f \in L^\infty(\mathbb{R}^d)$ and the right hand side being finite.

Method of proof

Let $g(t) = t(1+t^2)^{-1/2}$. Then $g(\mathcal{D})$ is a genuine pseudodifferential operator. First we show that,

$$\|[\operatorname{sgn}(\mathcal{D}) - g(\mathcal{D}), 1 \otimes M_f]\|_d \leq K_d' \|f\|_d$$

using Cwikel estimates Then we obtain the bound

$$\|[g(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} \leq K_d \|\nabla f\|_{L^d(\mathbb{R}^d, \mathbb{C}^d)}$$

using double operator integrals.

Using Cwikel estimates

For $j = 1, \ldots, d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ let

$$h_j(x) = \frac{x_j}{\|x\|} - \frac{x_j}{\sqrt{1 + \|x\|^2}}.$$

Then all we need to do is show that $\|h_j(-i\nabla)M_f\|_d \leq K_d'\|f\|_d$. From Cwikel estimates, this follows from $h_j \in L^d$, which is easily checked.

Double Operator Integrals

We also get the formula

$$[f(A), B] = \mathcal{T}^{A,A}_{f^{[1]}}([A, B])$$

when $[A, B] \in \mathcal{L}_2$.
Hence if we know that $[A, B] \in \mathcal{L}_p$, then to conclude that $[f(A), B] \in \mathcal{L}_p$ it is enough to show that $\mathcal{T}^{A,A}_{f^{[1]}}$ is bounded from $\mathcal{L}_p$ to $\mathcal{L}_p$.

Back to the proof

Our goal is to show that

$$\|[g(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} \leq K_d \|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}.$$

We start by considering the transfomer $\mathcal{T}_{g^{[1]}}^{\mathcal{D},\mathcal{D}}$.

The function g

Recall that $g(t) = t(1+t^2)^{-1/2}$. Then a whole lot of algebra shows that

$$g^{[1]}(\lambda,\mu) = \psi_1(\lambda,\mu)\psi_2(\lambda,\mu)\psi_3(\lambda,\mu).$$

Where,

$$\begin{aligned}
\psi_1(\lambda,\mu) &:= 1 + \frac{1-\lambda\mu}{(1+\lambda^2)^{1/2}(1+\mu^2)^{1/2}}\\
\psi_2(\lambda,\mu) &:= \frac{(1+\lambda^2)^{1/4}(1+\mu^2)^{1/4}}{(1+\lambda^2)^{1/2}+(1+\mu^2)^{1/2}}\\
\psi_3(\lambda,\mu) &:= \frac{1}{(1+\lambda^2)^{1/4}(1+\mu^2)^{1/4}}.
\end{aligned}$$

Note that ψ_1 and ψ_3 are in $L^\infty(\mathbb{R}) \otimes L^\infty(\mathbb{R})$.

On ψ_2

The following lemma is crucial:

Lemma

$\mathcal{T}^{\mathcal{D},\mathcal{D}}_{\psi_2}$ is bounded from $\mathcal{L}_{d,\infty}$ to $\mathcal{L}_{d,\infty}$.

Hence we have,

$$\begin{aligned}[g(\mathcal{D}), 1\otimes M_f] &= \mathcal{T}^{\mathcal{D},\mathcal{D}}_{g^{[1]}}([\mathcal{D}, 1\otimes M_f]) \\ &= \mathcal{T}^{\mathcal{D},\mathcal{D}}_{\psi_1}\mathcal{T}^{\mathcal{D},\mathcal{D}}_{\psi_2}((1+\mathcal{D}^2)^{-1/4}[\mathcal{D}, 1\otimes M_f](1+\mathcal{D}^2)^{-1/4}\end{aligned}$$

The Bikchantaev Conjecture

We have the following result.

Theorem (Sukochev, 2013)

Let $A > 0$ be an operator on H, and $B \in \mathcal{B}(H)$. Then for every $\theta \in (0,1)$, we have the submajorisation,

$$B^{\theta} A B^{1-\theta} << AB.$$

Hence,

$$\|B^{1/2} A B^{1/2}\|_{d,\infty} \lesssim \|AB\|_{d,\infty}.$$

Finishing the proof

Thus,

$$[g(\mathcal{D}), 1 \otimes M_f] << [\mathcal{D}, 1 \otimes M_f](1+\mathcal{D}^2)^{-1/2}.$$

Combining all our results so far, we get

$$\begin{aligned}\|\bar{d}f\|_{d,\infty} &\leq \|\bar{d}f - i[g(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} + \|i[g(\mathcal{D}), 1 \otimes M_f]\|_{d,\infty} \\ &\lesssim \|f\|_d + \|\nabla f\|_{L^d(\mathbb{R}^d, \mathbb{C}^d)}.\end{aligned}$$

Then we can use a "dilation trick" to eliminate the dependence on $\|f\|_d$.

Expanding the class of f

Everything that we have done so far relies on the assumption that $f \in \mathcal{S}(\mathbb{R}^d)$.
To extend to $f \in L^\infty(\mathbb{R}^d)$, we use the fact that for all $f \in L^\infty(\mathbb{R}^d)$ such that $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$ there exists a sequence $\{f_n\}_{n=0}^\infty \subset \mathcal{S}(\mathbb{R}^d)$ such that $f_n \to f$ uniformly on bounded sets and $\|\nabla f_n - \nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)}$ goes to 0.

Next Steps

1. Next we need to prove our "classical limit" trace formula,

$$\varphi(|\bar{d}f|^d) = c_d \int_{\mathbb{R}^d} \|\nabla f\|_2^d \, dx.$$

2. To then wrap everything up by proving that

$$\|\nabla f\|_{L^d(\mathbb{R}^d, \mathbb{C}^d)} \leq C_d \|\bar{d}f\|_{d,\infty}.$$

The constant c_d can be computed by following the argument carefully.

General Strategy of proof

Firstly assume that $f \in C_c^\infty(\mathbb{R}^d)$ for now (we will remove this assumption later). Furthermore assume that f is *real valued*.
It is hard to compute $\varphi(|\bar{d}f|^d)$ directly: $|\bar{d}f|^d$ is certainly not a pseudodifferential operator in general.
Instead we find a pseudodifferential operator A such that

$$\bar{d}f \in A(1+\mathcal{D}^2)^{-1/2} + \mathcal{L}_{2d/3,\infty}.$$

Quite general computations then lead us to:

$$\varphi(|\bar{d}f|^d) = \varphi(|A|^d(1+\mathcal{D}^2)^{-d/2})$$

which is a bit better.

Connes' trace formula

Recall that if σ is the symbol of a pseudodifferential operator T_σ and σ has an asymptotic expansion,

$$\sigma(x,\xi) \sim \sum_{j=-m}^{-\infty} p_{j,T_\sigma}(x,\xi)$$

as $\|\xi\| \to \infty$ and p_j is homogeneous of order j in ξ, the function p_{-m} is called the principal symbol of T_σ, and T_σ is of order $-m$. If such an asymptotic expansion exists, then T_σ is called *classical*.

Note that σ is matrix valued!

Properties of ΨDO's

A pseudodifferential operator T on $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^d)$ is said to be compactly based if there is $\phi \in C_c^\infty(\mathbb{R}^d)$ such that

$$(1 \otimes M_\phi)T = T$$

and compactly supported if there is also $\psi \in C_c^\infty$ such that

$$(1 \otimes M_\phi)T = T(1 \otimes M_\psi) = T.$$

More properties of ΨDO's

The principal symbol is multiplicative,

$$p_{n+m,AB} = p_{n,A}p_{m,B}.$$

if A and B are of order n and m respectively. This has some surprisingly strong consequences, such as:

Lemma

Let T be a compactly supported pseudodifferential operator of order $-\beta$. Then $T \in \mathcal{L}_{d/\beta,\infty}$.

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be such that $T(1\otimes M_\phi) = T$.
Then $T(1+\mathcal{D}^2)^{\beta/2}$ is bounded, so there is a bounded operator A such that

$$T = A(1+\mathcal{D}^2)^{-\beta/2}.$$

Hence $T = A(1+\mathcal{D}^2)^{-\beta/2}(1\otimes M_\phi)$ and so the result follows from Cwikel estimates.

A commutator lemma

F. Sukochev, E. McDonald

To warm up, we prove the following:

Lemma

Let A be a compactly based classical pseudodifferential operator with self-adjoint extension to $L^2(\mathbb{R}^d, \mathbb{C}^d)$. Then for all $\beta \geq 0$ and $0 < \alpha < d - 1$,

$$[(1+\mathcal{D}^2)^{-\alpha/2}, |A|](1+\mathcal{D}^2)^{-\beta/2} \in \mathcal{L}_{d/(\alpha+\beta+1),\infty}.$$

First we let $\phi \in C_c^\infty(\mathbb{R}^d)$ and we prove that,

$$[(1+\mathcal{D}^2)^{-\alpha/2}, 1\otimes M_\phi](1+\mathcal{D}^2)^{-\beta/2} \in \mathcal{L}_{d/(\alpha+\beta+1),\infty}.$$

Pick $\psi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi\psi = \phi$. Then,

$$[(1+\mathcal{D}^2)^{-\alpha/2}, (1\otimes M_\phi)(1\otimes M_\psi)](1+\mathcal{D}^2)^{-\beta/2} =$$
$$[(1+\mathcal{D}^2)^{-\alpha/2}, 1\otimes M_\phi](1\otimes M_\phi)(1+\mathcal{D}^2)^{-\beta/2}$$
$$+(1\otimes M_\phi)[(1+\mathcal{D}^2)^{-\alpha/2}, 1\otimes M_\psi)](1+\mathcal{D}^2)^{-\beta/2}.$$

So the result follows from our previous lemma and Cwikel estimates.

Now let A be compactly based, and $(1\otimes M_\psi)A = A$. Then,

$$[(1+\mathcal{D}^2)^{-\alpha/2},(1\otimes M_\psi)A](1+\mathcal{D}^2)^{-\beta/2} =$$
$$[(1+\mathcal{D}^2)^{-\alpha/2},1\otimes M_\phi]A(1+\mathcal{D}^2)^{-\beta/2}$$
$$+(1\otimes M_\phi)[(1+\mathcal{D}^2)^{-\alpha/2},A](1+\mathcal{D}^2)^{-\beta/2}.$$

So we get $[(1+\mathcal{D}^2)^{-\alpha/2},A](1+\mathcal{D}^2)^{-\beta/2}\in\mathcal{L}_{d/(\alpha+\beta+1),\infty}$. Then we get the final result from a DOI computation.

Connes' trace formula

The following is a consequence of Connes' trace theorem.

Theorem

Let T be a compactly based pseudodifferential operator of order 0 on $\mathcal{S}(\mathbb{R}^d,\mathbb{C}^d)$ with self-adjoint extension to $L^2(\mathbb{R}^d,\mathbb{C}^d)$ and principal symbol $p_{0,T}$. Then there is a constant $k_d>0$ such that for every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$,

$$\varphi(T(1+\mathcal{D}^2)^{-d/2}) = k_d\int_{\mathbb{R}^d}\int_{S^{d-1}}\mathrm{tr}(p_{0,T})(x,\xi)\,d\xi dx.$$

It turns out that $A(1+\mathcal{D}^2)^{-d/2}$ is a pseudodifferential operator with principal symbol $p_{0,A}(x,\xi)\frac{1}{\|\xi\|_2^d}$.
Then we simply use the fact that any trace φ on $\mathcal{L}_{1,\infty}\otimes M_N(\mathbb{C})$ splits as a product $\varphi = \varphi'\otimes \mathrm{tr}$.

Connes' trace formula, cont.

Recall that we wanted to compute a trace like $\varphi(|A|^d(1+\mathcal{D}^2)^{-d/2})$.
Since principal symbols are multiplicative, we easily get that for every polynomial h with $h(0)=0$,

$$\varphi(h(A)(1+\mathcal{D}^2)^{-d/2}) = k_d\int_{\mathbb{R}^d}\int_{S^{d-1}}\mathrm{tr}(h(p_{0,A}(x,\xi)))\,d\xi dx.$$

The above is also true for continuous functions h, and the following can be shown by approximating h uniformly with polynomials:

Lemma

If h is continuous on the spectrum of A, and $h(0)=0$, then

$$\varphi(h(A)(1+\mathcal{D}^2)^{-d/2}) = k_d\int_{\mathbb{R}^d}\int_{S^{d-1}}\mathrm{tr}(h(p_{0,A}(x,\xi)))(x,\xi)\,d\xi dx.$$

Finding a good candidate for A

Now what we need to do is find some good compactly based self-adjoint pseudodifferential operator A of order 0 such that

$$\bar{d}f \in A(1+\mathcal{D}^2)^{-1/2} + \mathcal{L}_{2d/3,\infty}.$$

Replacing sgn with g

$g(\mathcal{D})$ is a pseudodifferential operator. It is much nicer to work with $i[g(\mathcal{D}), 1\otimes M_f]$ than $\bar{d}f$.
Fortunately, we have the following:

$$(g(\mathcal{D}) - \mathrm{sgn}(\mathcal{D}))(1\otimes M_f) \in \mathcal{L}_{d/2}$$

from a previous talk.
So it's enough to find A such that

$$i[g(\mathcal{D}), 1\otimes M_f] \in A(1+\mathcal{D}^2)^{-1/2} + \mathcal{L}_{2d/3,\infty}.$$

Defining A

We can in fact do better.
Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be arbitrary. For $k = 1, 2, \ldots, d$, define

$$A_k := M_{\partial_k f} - \frac{1}{2}\sum_{j=1}^{d} M_\phi \frac{D_j D_k}{1-\Delta} M_{\partial_j f} + M_\phi \frac{D_j D_k}{1-\Delta} M_{\partial_j f}$$

and then,

$$A := \sum_{k=1}^{d} \gamma_k \otimes A_k.$$

With the above definition of A

Theorem

$$i[g(\mathcal{D}), 1 \otimes M_f] \in A(1+\mathcal{D}^2)^{-1/2} + \mathcal{L}_{d/2,\infty}.$$

(Proof omitted, for length and for being uninteresting)

Taking the trace

Now we prove the key result of this section, which is actually a little more general than needed.

Lemma

If T is an operator on $L^2(\mathbb{R}^d, \mathbb{C}^d)$ and A is a compactly supported self-adjoint pseudodifferential operator of order 0 with self-adjoint bounded extension and

$$T \in A(1+\mathcal{D}^2)^{-1/2} + \mathcal{L}_{2d/3,\infty}$$

then

$$\varphi(|T|^d) = \varphi(|A|^d(1+\mathcal{D}^2)^{-d/2}).$$

Let $K = (1+\mathcal{D}^2)^{-1/2}$.
So we have $T - AK \in \mathcal{L}_{2d/3,\infty}$. Hence, $T^* - KA \in \mathcal{L}_{2d/3,\infty}$. So we get,

$$|T|^2 + K|A|^2K - KAT - T^*AK \in \mathcal{L}_{d/3,\infty}.$$

So,

$$|T|^2 - K|A|^2K \in KA(AK - T) + (KA - T^*)AK + \mathcal{L}_{d/3,\infty}.$$

So, since $AK \in \mathcal{L}_{d,\infty}$,

$$|T|^2 - K|A|^2K \in \mathcal{L}_{2d/5,\infty}.$$

Now we split the proof into $d = 2$ and $d > 2$.
For $d = 2$, we are done since $\mathcal{L}_{4/5,\infty} \subset \mathcal{L}_1$.
Now suppose $d > 2$.
Now,

$$[K^\alpha, |A|]K^\beta \in \mathcal{L}_{d/(\alpha+\beta+1),\infty}.$$

So setting $\alpha = \beta = 1/2$,

$$K|A| \in K^{1/2}|A|K^{1/2} \in \mathcal{L}_{d/2,\infty}.$$

So,

$$K|A|^2K \in (K^{1/2}|A|K^{1/2})^2 + \mathcal{L}_{d/3,\infty}.$$

So putting this all together,

$$|T|^2 \in (K^{1/2}|A|K^{1/2})^2 + \mathcal{L}_{2d/5,\infty}.$$

So using the Birman-Koplienko-Solomyak formula,

$$|T| \in K^{1/2}|A|K^{1/2} + \mathcal{L}_{4d/5,\infty}.$$

So,

$$|T| \in |A|K + \mathcal{L}_{4d/5,\infty}.$$

It then follows that,

$$|T|^d \in (|A|K)^d + \mathcal{L}_1.$$

So far we have established that

$$\varphi(|T|^d) = \varphi((|A|K)^d).$$

So we will be done if we can show that

$$(|A|K)^d \in |A|^{d-2}K^d|A|^2 + \mathcal{L}_1.$$

This is shown by a fairly convoluted induction on j, that for all $0 \leq j \leq d-2$,

$$(|A|K)^d \in |A|^j K^j (|A|K)^{d-j} + \mathcal{L}_1.$$

Nearing the end

Now we can apply Connes' trace theorem to get:

$$\varphi(|\bar{d}f|^d) = \int_{\mathbb{R}^d}\int_{S^{d-1}} \mathrm{tr}(|p_{0,A}(x,\xi)|^d)\,dxd\xi.$$

We can then compute p_{0,A_k}:

$$p_{0,A_k}(x,\xi) = \partial_k f(x) - \sum_{j=1}^d \phi(x)\partial_j f(x)\frac{\xi_k\xi_j}{\|\xi\|^2}$$

And so in fact:

$$\mathrm{tr}(|p_{0,A}(x,\xi)|^d) = \left\|\nabla f(x) - \frac{\xi(\xi,\nabla f)}{\|\xi\|^2}\right\|_2^d.$$

So then we get:

$$\varphi(|\bar{d}f|^d) = k_d \int_{\mathbb{R}^d} \int_{S^{d-1}} \|\nabla f(x)\|^d \left\| \frac{\nabla f}{\|\nabla f\|_2} - \xi \left(\xi, \frac{\nabla f}{\|\nabla f\|_2} \right) \right\|_2^d dx d\xi.$$

Since the inner integral is rotationally invariant,

$$\varphi(|\bar{d}f|^d) = k_d \int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d \, dx \int_{S^{d-1}} \|e_1 - \xi\xi_1\|_2^d \, d\xi.$$

So set $c_d = k_d \int_{S^{d-1}} \|e_1 - \xi\xi_1\|_2^d \, d\xi$ and we are done.

We're not really done

So far we have proved that

$$\varphi(|\bar{d}f|^d) = c_d \int_{\mathbb{R}^d} \|\nabla f(x)\|_2^d \, dx$$

under the assumption that $f \in C_c^\infty(\mathbb{R}^d)$. We can do better.

Broadening the class of possible f

Now we can prove the trace formula for $f \in L^\infty(\mathbb{R}^d)$ such that $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$.
To do this: note that we can choose $\{f_n\}_{n=0}^\infty \subset C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ uniformly on compact sets.
Hence, $đ f_n$ converges to $đ f$ in the strong operator topology, and $\{đ f_n\}_{n=0}^\infty$ is Cauchy in the $\mathcal{L}_{d,\infty}$ quasinorm.

Finishing things off

So we have proved that $\varphi(|\bar{d}f|^d) = c_d \int_{\mathbb{R}^d} \|\nabla f\|_2^d \, dx$ when $f \in L^\infty(\mathbb{R}^d)$ and $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$.
To be really thoroughly finished, we should show that the equality holds when either side of the equation is defined. Thus we wish to prove that if $f \in L^\infty(\mathbb{R}^d)$ is such that $\bar{d}f \in \mathcal{L}_{d,\infty}$, then $\nabla f \in L^d(\mathbb{R}^d, \mathbb{C}^d)$.
Specifically, we will prove that

$$\|\nabla f\|_{L^d(\mathbb{R}^d, \mathbb{C}^d)} \leq C_d \|\bar{d}f\|_{d,\infty}$$

For some constant $C_d > 0$.

Using the trace formula

Note that we already have:

$$\|\nabla f\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \le C_d \|\bar{d}f\|_{d,\infty}$$

for f *a priori* such that $\nabla f \in L^d(\mathbb{R}^d,\mathbb{C}^d)$.

Using Banach-Alaoglu

We use the following common sort of argument in the theory of distributions:

Lemma

Let $f \in \mathcal{S}'(\mathbb{R}^d)$ be a tempered distribution, and let $\{\phi_j\}_{j=1}^d$ be an approximate identity of C_c^∞ function, and $\psi_n(t) = \psi(t/n)$ is a sequence of cut-off functions. Let $1 < p < \infty$. If,

$$\sup_{n\geq 0} \|\nabla((\phi_n * f)\psi_n))\|_p < \infty$$

then $\nabla f \in L^p(\mathbb{R}^d, \mathbb{C}^d)$.

Basically:
The sequence $\nabla(\phi_n * f)\psi_n$ is bounded in L^p, which is reflexive for $1 < p < \infty$. So by the Banach-Alaoglu theorem there is a weakly converging subsequence to some element of L^p. General distributional theory implies then that the limit point is ∇f.

With this in mind, let $\{\phi_n\}_{n=0}^\infty$ and $\{\psi_n\}_{n=0}^\infty$ be as above.
If $f \in L^\infty(\mathbb{R}^d)$, then $(f * \phi_n)\psi_n \in C_c^\infty(\mathbb{R}^d)$.
So we have:

$$\|\nabla((f * \phi_n)\psi_n)\|_{L^d(\mathbb{R}^d,\mathbb{C}^d)} \leq C_d \|đ((f * \phi_n)\psi_n)\|_{d,\infty}.$$

Straightforward computations show that the right hand side is bounded, and so we are (essentially done).

Future Prospects

Conjecture

If (A, H, D) is a p-summable spectral triple and $a \in A$, and φ is a continuous normalised trace,

$$\varphi(|[\mathrm{sgn}(\mathcal{D}), a]|^p) = c\varphi(|[\mathcal{D}, a]|^p(1+\mathcal{D}^2)^{-p/2}). \tag{1}$$

The End.

Thank you for listening!