Double operator integrals and applications

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Operator integrals on finite matrices

Let $n \in \mathbb{N}$ and let $\mathcal{B}(\mathbb{C}^n)$ be the space of all linear operators on $\mathbb{C}^n$.

Suppose that A and B are selfadjoint linear operators on $\mathbb{C}^n$ with respective complete systems of orthonormal eigenvectors

$$\{\xi_j\}_{j=1}^n, \quad \{\eta_k\}_{k=1}^n$$

and eigenvalues

$$\{\lambda_j\}_{j=1}^n, \quad \{\mu_k\}_{k=1}^n.$$

We denote by P_ξ the orthogonal projection on the unit vector $\xi \in \mathbb{C}^n$, that is $P_\xi = \langle \cdot, \xi \rangle \xi$.

Operator integrals on finite matrices

For a function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be a complex valued function and $X \in \mathcal{B}(\mathbb{C}^n)$ we denote

$$T_\varphi^{A,B}(X) := \sum_{j=1}^{n} \sum_{k=1}^{n} \varphi(\lambda_j, \mu_k) P_{\xi_j} X P_{\eta_k}, \tag{1}$$

Note that the expression (1) does not depend on the order of eigenvectors chosen.

The mapping

$$T_\varphi^{A,B} : \mathcal{B}(\mathbb{C}^n) \to \mathcal{B}(\mathbb{C}^n)$$

is a discrete version of a double operator integral and the function φ is usually called the symbol of the operator $T_\varphi^{A,B}$.

Matrix representation of $T_{\varphi}^{A,B}(X)$

As a bounded operator on $\mathbb{C}^n$, the operator $T_{\varphi}^{A,B}(X)$ can be identified with $n \times n$ matrix with respect to $\{\xi_j\}_{j=1}^n$ and $\{\eta_k\}_{k=1}^n$.

Let $X \in \mathcal{B}(\mathbb{C}^n)$ is identified with the $n \times n$ complex matrix

$$X = (x_{jk})_{j,k=1}^n = \{\langle X\eta_k, \xi_j\rangle\}_{j,k=1}^n.$$

Since the systems $\{\xi_j\}_{j=1}^n$, $\{\eta_k\}_{k=1}^n$ are orthonormal, for $1 \le j_0, k_0 \le n$ we can compute

$$\begin{aligned}\langle T_{\varphi}^{A,B}(X)(\eta_{k_0}), \xi_{j_0}\rangle &= \sum_{j=1}^n \sum_{k=1}^n \varphi(\lambda_j, \mu_k) \Big\langle P_{\xi_j} X P_{\eta_k}(\eta_{k_0}), \xi_{j_0} \Big\rangle \\ &= \sum_{j=1}^n \varphi(\lambda_j, \mu_k) \langle P_{\xi_j} X \eta_{k_0}, \xi_{j_0}\rangle \\ &= \sum_{j=1}^n \varphi(\lambda_j, \mu_k) \langle X\eta_{k_0}, P_{\xi_j}\xi_{j_0}\rangle = \varphi(\lambda_{j_0}, \mu_{k_0}) x_{j_0 k_0}. \qquad (2)\end{aligned}$$

Thus, the matrix representation of $T_{\varphi}^{A,B}(X)$ is given by

$$T_{\varphi}^{A,B}(X) = \{\varphi(\lambda_j, \mu_k) x_{jk}\}_{j,k=1}^{n}.$$

That is, the operator $T_{\varphi}^{A,B}(X)$ is the pointwise product (usually called Schur product) of matrices $\{\varphi(\lambda_j, \mu_k)\}_{j,k=1}^{n}$ and $\{x_{jk}\}_{j,k=1}^{n}$.

The double operator integral $T_{\varphi}^{A,B}$ (as well as its infinite dimensional analogue) has a lot of fine properties, which we will cover later. It is extremely powerful tool to study various problems in operator theory.

We demonstrate the solution (in the case of finite matrices) of few the most interesting problems that may be resolved via double operator integration techniques. The first problem is that of Lipschitz estimates for operator-valued functions and the second is differentiation of operator functions.

Lipschitz estimate

Let $\|\cdot\|_2$ by the Hilbert-Schmidt norm on $\mathcal{B}(\mathbb{C}^n)$, that is

$$\|X\|_2 = \Big(\sum_{j,k=1}^{n} |x_{jk}|^2\Big)^{1/2}, \quad X = \{x_{jk}\}_{j,k=1}^{n} \in \mathcal{B}(\mathbb{C}^n).$$

Using representation of $T_\varphi^{A,B}(X)$ as the Schur product of matrices $\{\varphi(\lambda_j, \mu_k)\}_{j,k=1}^{n}$ and $\{x_{jk}\}_{j,k=1}^{n}$, we can estimate

$$\|T_\varphi^{A,B}(X)\|_2^2 = \sum_{j,k=1}^{n} |\varphi(\lambda_j, \mu_k)|^2 |x_{jk}|^2 \leq \max_{j,k} |\varphi(\lambda_j, \mu_k)|^2 \ \|X\|_2^2.$$

Hence

$$\|T_\varphi^{A,B}(X)\|_2 \leq \max_{j,k} |\varphi(\lambda_j, \mu_k)| \ \|X\|_2. \tag{3}$$

An estimate of type (3) is the best possible. We shall always strive to obtain analogous estimates for double operator integrals for other norms on $\mathcal{B}(\mathbb{C}^n)$ and its infinite dimensional analogues.

Lipschitz estimate

Let f be a real-valued function defined on a segment $[a, b] \subseteq \sigma(A) \cup \sigma(B)$, where $\sigma(X)$ is the spectrum of the matrix X. Let $\mathrm{Lip}(a, b)$ be the class of all Lipschitz functions on $[a, b]$ with the norm

$$\|f\|_{Lip} := \sup_{\substack{t,s \in [a,b] \\ t \neq s}} \frac{f(t) - f(s)}{t - s}.$$

Consider the divided difference $f^{[1]}$ given by

$$f^{[1]}(\lambda_j, \mu_k) := \begin{cases} \frac{f(\lambda_j) - f(\mu_k)}{\lambda_j - \mu_k}, & \lambda_j \neq \mu_k \\ f'(\lambda_j), & \lambda_j = \mu_k \end{cases}, \quad 1 \leq j, k \leq n.$$

The discrete symbol $f^{[1]}$ and the corresponding double operator integral was first studied by Löwner in 1934, where he noted that since

$$f(A)\xi_j = f(\lambda_j)\xi_j, \quad f(B)\eta_k = f(\mu_k)\eta_k,$$

we have

$$\begin{aligned}
\Big\langle (f(A) - f(B))\eta_k, \xi_j \Big\rangle &= \langle \eta_k, f(A)\xi_j \rangle - \langle f(B)\eta_k, \xi_j \rangle \\
&= \big(f(\lambda_j) - f(\mu_k)\big)\langle \eta_k, \xi_j \rangle \\
&= \frac{f(\lambda_j) - f(\mu_k)}{\lambda_j - \mu_k} \langle (A - B)\eta_k, \xi_j \rangle \\
&= f^{[1]}(\lambda_j, \mu_k)\Big\langle (A - B)\eta_k, \xi_j \Big\rangle.
\end{aligned}$$

Thus the Löwner's formula can be written in terms of a double operator integral as

$$f(A) - f(B) = T^{A,B}_{f^{[1]}}(A - B).$$

Hence, we have that

$$\|f(A)-f(B)\|_2 = \|T_{f^{[1]}}^{A,B}(A-B)\|_2 \leq \max_{j,k}|f^{[1]}(\lambda_j,\mu_k)|\|A-B\|_2.$$

It is well-known that if f is a Lipschitz function, then

$$\max_{j,k}|f^{[1]}(\lambda_j,\mu_k)| \leq \|f\|_{Lip}.$$

Thus, we conclude that

$$\|f(A)-f(B)\|_2 \leq \|f\|_{Lip}\|A-B\|_2$$

for all $A=A^*, B=B^* \in \mathcal{B}(\mathbb{C}^n)$.

Differentiation of operator functions

The first paper addressing the issue of differentiation of operator functions is due to Daleckii and Krein, 1956.

Let $X(t)$ be a selfadjoint $n \times n$ matrix with eigenvalues $\{\lambda_j(t)\}_{j=1}^n$ and respective eigenvectors $\{\xi_j(t)\}_{j=1}^n$, depending on the parameter t and let f be some function on $\mathbb{R}$.

Suppose that the entries of the matrix $X(t)$ are differentiable functions with respect to the parameter t.

The problem which is of interest to us here is to estimate the norm of the derivative of the operator-valued function

$$t \to f(X(t)).$$

Consider firstly the case when

$$f(\lambda) = \lambda^m, \quad m \in \mathbb{N}.$$

Then, differentiating by the product rule (and suppressing the parameter t)

$$\begin{aligned}
\frac{df(X(t))}{dt} = \frac{dH^m(t)}{dt} &= \sum_{i=0}^{m-1} H^i \frac{dX(t)}{dt} H^{m-i-1} \\
&= \sum_{i=0}^{m-1} \sum_{j=1}^{n} \lambda_j^i P_{\xi_j} \frac{dX(t)}{dt} \sum_{k=1}^{n} \lambda_k^{m-i-1} P_{\xi_k} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \Big(\sum_{i=0}^{m-1} \lambda_j^i \lambda_k^{m-i-1} \Big) P_{\xi_j} \frac{dX(t)}{dt} P_{\xi_k} \\
&= \begin{cases} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\lambda_j^m - \lambda_k^m}{\lambda_j - \lambda_k} P_{\xi_j} \frac{dX(t)}{dt} P_{\xi_k}, & \lambda_j \neq \lambda_k \\ \sum_{j=1}^{n} \sum_{k=1}^{n} m\lambda_j^{m-1} P_{\xi_j} \frac{dX(t)}{dt} P_{\xi_k}, & \lambda_j = \lambda_k \end{cases} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} f^{[1]}(\lambda_j, \lambda_k) P_{\xi_j} \frac{dX(t)}{dt} P_{\xi_k} = T_{f^{[1]}}^{X(t),X(t)} \Big(\frac{dX(t)}{dt} \Big).
\end{aligned}$$

By linearity the formula above also holds for polynomials.
Hence,approximating an arbitrary differentiable function f by polynomials we obtain

$$\frac{df(X(t))}{dt} = T_{f^{[1]}}^{X(t),X(t)}\Big(\frac{dX(t)}{dt}\Big).$$

This representation of the derivative of $f(X(t))$ allows us to conclude that

$$\Big\|\frac{df(X(t))}{dt}\Big\|_2 = \|T_{f^{[1]}}^{X(t),X(t)}\Big(\frac{dX(t)}{dt}\Big)\|_2 \leq \max_{a\leq\lambda\leq b}|f'(\lambda)|\Big\|\frac{dX(t)}{dt}\Big\|_2.$$

Before we proceed to the continuous analogue of double operator integrals we make the following useful observation.
Assume that the function φ has the representation

$$\varphi(\lambda, \mu) = a_1(\lambda) a_2(\mu)$$

for some complex valued functions a_1 and a_2. The definition of double operator integral and the spectral resolution of operator functions $a_1(A)$ and $a_2(B)$ imply that

$$T_\varphi^{A,B}(X) = a_1(A) \cdot X \cdot a_2(B), \quad X \in \mathcal{B}(\mathbb{C}^n).$$

The preceding formula lies at the foundation of the definition of continuous variant of operator integral.

Operator integrals on Schatten classes and $\mathcal{B}(\mathcal{H})$

Let $\mathcal{H}$ be a Hilbert space equipped with the norm $\|\cdot\|_{\mathcal{H}}$ and let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra of all bounded linear operators on $\mathcal{H}$ equipped with the uniform norm $\|X\|_{\mathcal{B}(\mathcal{H})}$.
Let $1 \leq p < \infty$. By $\mathcal{S}_p$ we denote the Schatten (von Neumann) classes, that is the ideal in $\mathcal{B}(\mathcal{H})$ of all compact operators X on a Hilbert space $\mathcal{H}$, such that

$$\|X\|_p := (\mathrm{Tr}(|X|^p))^{1/p} < \infty,$$

where Tr is the standard trace on $\mathcal{B}(\mathcal{H})$. By $\mathcal{S}_\infty$ we denote the Banach space of all compact operators on $\mathcal{H}$ equipped with the uniform norm.
There are several definitions of the double operator integral on $\mathcal{S}_p$. We will present the simplest one.

Let $A, B \in \mathcal{B}(\mathcal{H})$ be self-adjoint operators and let $\varphi \in L^\infty(\mathbb{R} \times \mathbb{R})$ be given by the formula

$$\varphi(\lambda, \mu) = \sum_{j=1}^{n} a_1(j, \lambda) a_2(j, \mu), \quad \lambda, \mu \in \mathbb{R},$$

where $a_1(j, \cdot), a_2(j, \cdot) \in L^\infty(\mathbb{R})$. Define the operator $T_\varphi^{A,B}(X)$ as follows

$$T_\varphi^{A,B}(X) = \sum_{j=1}^{n} a_1(j, A) \cdot X \cdot a_2(j, B) \in \mathcal{S}_p.$$

Since $\mathcal{S}_p$ is a two-sided ideal in $\mathcal{B}(\mathcal{H})$, it is easy to see that $T_\varphi^{A,B}$ is a bounded linear operator on $\mathcal{S}_p$ for $1 \le p \le \infty$ and

$$\| T_\varphi^{A,B} \|_{\mathcal{S}_p \to \mathcal{S}_p} \le \sum_{j=1}^{n} \| a_1(j, \cdot) \|_\infty \| a_2(j, \cdot) \|_\infty .$$

Similarly, the operator $T_\varphi^{A,B}$ is defined on $\mathcal{B}(\mathcal{H})$.

Next we present the largest class of functions, for which this approach does work.
Let $\mathfrak{A}$ be the class of functions $\varphi : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}$ admitting the representation

$$\varphi(\lambda, \mu) = \int_\Omega a_1(\lambda, \omega)\, a_2(\mu, \omega) d\nu(\omega), \tag{4}$$

for some finite measure space (Ω, ν) and bounded Borel functions

$$a_j(\cdot, \omega) : \mathbb{R} \mapsto \mathbb{C}, \quad j = 1, 2.$$

The class $\mathfrak{A}$ is in fact an algebra with respect to the operations of pointwise addition and multiplication. The formula

$$\|\varphi\|_{\mathfrak{A}} = \inf \int_\Omega \|a_1(\cdot, \omega)\|_\infty \|a_2(\cdot, \omega)\|_\infty \, d|\nu|(\omega),$$

where the infimum is taken over all possible representations (4) defines a norm on $\mathfrak{A}$.

Next we define the double operator integral associated with the function $\varphi \in \mathfrak{A}$.

Let $1 \leq p \leq \infty$. For every $\varphi \in \mathfrak{A}$, and a fixed couple of self-adjoint operators A, B, the double operator integral $T_{\varphi}^{A,B} : \mathcal{S}^p \to \mathcal{S}^p$, is defined as follows

$$T_{\varphi}^{A,B}(X) = \int_{\Omega} a_1(A,\omega)\, X\, a_2(B,\omega) d\nu(\omega), \quad X \in \mathcal{S}^p,$$

where a_j's and (Ω, ν) are taken from the representation (4) and the integral is understood in the sense of the Bochner integral. The operator $T_{\varphi}^{A,B}$ on $\mathcal{B}(\mathcal{H})$ is defined similarly.

Theorem 1

1. *The definition of* $T_{\varphi}^{A,B}(X)$ *does not depend on the representation of the function* φ.
2. *The operator* $T_{\varphi}^{A,B}$ *is linear bounded on* $\mathcal{S}_p$, $1 \leq p \leq \infty$, *and on* $\mathcal{B}(\mathcal{H})$. *Moreover,*
$$\left\| T_{\varphi}^{A,B} \right\| \leq \|\varphi\|_{\mathfrak{A}} .$$

Lipschitz estimate

Similarly to the case of finite matrices, the double operator integral can be used to describe the difference $f(A) - f(B)$ for operators $A, B \in \mathcal{B}(\mathcal{H})$.

Theorem 2

Let $A = A^*, B = B^* \in \mathcal{B}(\mathcal{H})$. If f be a Lipschitz function on the segment $[a, b] \subseteq \sigma(A) \cup \sigma(B)$ such that $f^{[1]} \in \mathfrak{A}$, then we have

(i)

$$f(A) - f(B) = T^{A,B}_{f^{[1]}}(A - B);$$

(ii) $f(A) - f(B) \in \mathcal{S}_p$ *whenever* $A - B \in \mathcal{S}_p$, $1 \leq p \leq \infty$. *Moreover,*

$$\|f(A) - f(B)\|_p \leq \|f^{[1]}\|_{\mathfrak{A}} \|A - B\|_p.$$

The class of Lipschitz functions such that $f^{[1]} \in \mathfrak{A}$

Now we are interested in the question, for which functions $f : \mathbb{R} \to \mathbb{C}$ its divided difference $f^{[1]}$ belongs to the class $\mathfrak{A}$.
In Lemma 3 below we provide examples of such functions.
Let $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ be the Fourier transform and the inverse Fourier transform, respectively of the function $f : \mathbb{R} \to \mathbb{C}$, i.e.

$$\mathcal{F}f(t) = \int_{\mathbb{R}} f(s)\, e^{-ist}\, ds, \quad \mathcal{F}^{-1}f(s) = \int_{\mathbb{R}} f(t)\, e^{ist}\, dt.$$

By $W_1(\mathbb{R})$ we denote the class of functions $f : \mathbb{R} \to \mathbb{C}$ such that $\mathcal{F}f' \in L_1(\mathbb{R})$.

Lemma 3

If $f \in W_1(\mathbb{R})$, then $f^{[1]} \in \mathfrak{A}$.

Lipschitz estimates in $\mathcal{S}_p, 1 < p < \infty$

However, the restriction on the function f that $f^{[1]} \in \mathfrak{A}$, does not resolve completely the question whether a Lipschitz function is operator Lipschitz. However, another more subtle approach to DOI due to D. Potapov, F. Sukochev (which we will not introduce here) of double operator integral resolves this question.

Theorem 4 (Potapov-S, Acta Math. 2011)

Let f be a Lipschitz function on $\mathbb{R}$. Then there is a constant $C > 0$ such that

$$\|f(A) - f(B)\|_p \leq C\|A - B\|_p$$

holds for all $A = A^*, B = B^*$ with $A - B \in \mathcal{S}_p$ if and only if $1 < p < \infty$.

Lipschitz estimates for $\mathcal{S}_1$

In contrast to the case $1 < p < \infty$, there are examples Lipschitz functions on $\mathbb{R}$, which are NOT operator Lipschitz in the trace class ideal $\mathcal{S}_1$. For example, for the Lipschitz function $f = |\cdot|$, there exist $A, B \in \mathcal{B}(\mathcal{H})$ such that $A - B \in \mathcal{S}_1$ but

$$f(A) - f(B) = |A| - |B| \notin S_1.$$

Let A be compact operator and let $\{\mu_k(A)\}_{k\in\mathbb{N}}$ be the sequence of singular values of A, that is the sequence of eigenvalues of $|A|$ in decreasing order counting multiplicities. Define

$$\mathcal{S}_{1,\infty} = \{A \in S_\infty : \|A\|_{1,\infty} := \sup_n n \cdot \mu_n(A) < \infty\}.$$

Lipschitz estimates for $\mathcal{S}_1$

Theorem 5 (Caspers, Potapov, Sukochev, Zanin)

Let $A = A^*$ and $B = B^*$ be such that $A - B \in \mathcal{S}_1$ and f be a Lipschitz function on $\mathbb{R}$. Then $f(A) - f(B) \in \mathcal{S}_{1,\infty}$ and

$$\|f(A) - f(B)\|_{1,\infty} \leq \mathrm{const}\|A - B\|_1,$$

for some absolute constant.