Sidon sets in bounded orthonormal systems

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Sidon sets
Classical definitions:

$\Lambda \subset \mathbb{Z}$ is Sidon if

$$\sum_{n\in\Lambda} a_n e^{int} \in C(\mathbf{T}) \Rightarrow \sum_{n\in\Lambda} |a_n| < \infty$$

$\Lambda \subset \mathbb{Z}$ is randomly Sidon if

$$\sum_{n\in\Lambda} \pm a_n e^{int} \in C(\mathbf{T}) \ a.s. \Rightarrow \sum_{n\in\Lambda} |a_n| < \infty$$

$\Lambda \subset \mathbb{Z}$ is subGaussian if

$$\sum_{n\in\Lambda} |a_n|^2 < \infty \Rightarrow \int \exp |\sum_{n\in\Lambda} a_n e^{int}|^2 < \infty.$$

They are all equivalent !

Obviously Sidon $\Rightarrow$ randomly Sidon

Rudin (1961): Sidon $\Rightarrow$ subGaussian

Rider (1975) : Sidon $\Leftrightarrow$ randomly Sidon

P (1978) : Sidon $\Leftrightarrow$ subGaussian

Results hold more generally for any subset $\Lambda \subset \widehat{G}$
when G is any compact Abelian group

They are all equivalent !

Obviously Sidon $\Rightarrow$ randomly Sidon

Rudin (1961): Sidon $\Rightarrow$ subGaussian

Rider (1975) : Sidon $\Leftrightarrow$ randomly Sidon
(Note: This refines Drury's celebrated 1970 union Theorem)

P (1978) : Sidon $\Leftrightarrow$ subGaussian

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Examples

Hadamard lacunary sequences

$$n_1 < n_2 < \cdots < n_k, \cdots$$

such that

$$\inf_k \frac{n_{k+1}}{n_k} > 1$$

Explicit example

$$n_k = 2^k$$

Basic Example: Quasi-independent sets

Λ is quasi-independent if all the sums

$$\{\sum_{n\in A} n \mid A \subset \Lambda, |A| < \infty\}$$

are distinct numbers

$$\text{quasi-independent} \Rightarrow \text{Sidon}$$

The recent rebirth

Bourgain and Lewko (arxiv 2015) wondered whether a group environment is needed for all the preceding

Question

What remains valid if $\Lambda \subset \widehat{G}$ is replaced by a *uniformly bounded* orthonormal system ?

Let $\Lambda = \{\varphi_n\} \subset L_\infty(T, m)$ orthonormal in $L_2(T, m)$ ((T, m) any probability space)

(i) We say that (φ_n) is Sidon with constant C if for any n and any complex sequence (a_k) we have

$$\sum_1^n |a_k| \leq C \| \sum_1^n a_k \varphi_k \|_\infty .$$

(ii) We say that (φ_n) is randomly Sidon with constant C if for any n and any complex sequence (a_k) we have

$$\sum_1^n |a_k| \leq C \text{Average}_{\pm 1} \| \sum_1^n \pm a_k \varphi_k \|_\infty ,$$

(iii) Let $k \geq 1$. We say that (φ_n) is $\otimes^k$-Sidon with constant C if the system $\{\varphi_n(t_1) \cdots \varphi_n(t_k)\}$ (or equivalently $\{\varphi_n^{\otimes k}\}$) is Sidon with constant C in $L_\infty(T^k, m^{\otimes k})$.

Now assume merely that $\{\varphi_n\} \subset L_2(T, m)$.

(iv) We say that (φ_n) is subGaussian with constant C (or C-subGaussian) if for any n and any complex sequence (a_k) we have

$$\| \sum_1^n a_k \varphi_k \|_{\psi_2} \leq C (\sum |a_k|^2)^{1/2} .$$

Here

$$\psi_2(x) = \exp x^2 - 1$$

and $\|f\|_{\psi_2}$ is the norm in associated Orlicz space

Again: We say that $\{\varphi_n\} \subset L_2(T, m)$ is subGaussian with constant C (or C-subGaussian) if for any n and any complex sequence (a_k) we have

$$\| \sum_1^n a_k \varphi_k \|_{\psi_2} \leq C (\sum |a_k|^2)^{1/2}.$$

Equivalently, assuming w.l.o.g. $\int \varphi_k = 0, \forall k$
$\exists C$ such that $\forall (a_k)$

$$\int \exp Re(\sum_1^n a_k \varphi_k) \leq \exp C^2 \sum |a_k|^2$$

Important remark: Standard i.i.d. (real or complex) Gaussian random variables are subGaussian (Fundamental example !)

Easy Observation : *Sidon* $\not\Rightarrow$ *subGaussian*

By a much more delicate example Bourgain and Lewko proved:

$$subGaussian \not\Rightarrow Sidon$$

However, they proved

Theorem

$$subGaussian \Rightarrow \otimes^5 - Sidon$$

Recall $\otimes^5 - Sidon$ *means*

$$\sum_1^n |a_k| \leq C \| \sum_1^n a_k \varphi_k(t_1) \cdots \varphi_k(t_5) \|_{L_\infty(T^5)}.$$

This generalizes my 1978 result that subGaussian implies Sidon
for characters $(\varphi_k(t_1) \cdots \varphi_k(t_5) = \varphi_k(t_1 \cdots t_5)$!)
They asked whether 5 can be replaced by 2 which would be optimal

Indeed, it is so.

Theorem

For bounded orthonormal systems

$$subGaussian \Rightarrow \otimes^2 - Sidon$$

Recall $\otimes^2 - Sidon$ *means*

$$\sum_1^n |a_k| \le C \| \sum_1^n a_k \varphi_k(t_1)\varphi_k(t_2)\|_{L_\infty(T^2)}.$$

Actually, we have more generally:

Theorem (1)

Let $(\psi_n^1), (\psi_n^2)$ be systems biorthogonal respectively to $(\varphi_n^1), (\varphi_n^2)$ on probability spaces $(T_1, m_1), (T_2, m_2)$ resp. and uniformly bounded respectively by C_1', C_2',
If $(\varphi_n^1), (\varphi_n^2)$ are subGaussian with constants C_1, C_2 then

$$\sum |a_n| \leq \alpha \ \mathrm{ess\,sup}_{(t_1,t_2)\in T_1\times T_2} \left| \sum a_n \psi_n^1(t_1)\psi_n^2(t_2) \right|,$$

where α is a constant depending only on C_1, C_2, C_1', C_2'.

To illustrate by a concrete (but trivial) example: take $\varphi_n^1 = \varphi_n^2 = g_n$ and $\psi_n^1 = \psi_n^2 = \mathrm{sign}(g_n)$

The key new ingredient is a corollary of a powerful result due to **Talagrand Acta (1985)** (combined with a soft Hahn-Banach argument)
Let (g_n) be an i.i.d. sequence of standard (real or complex) Gaussian random variables

Theorem

Let (φ_n) be C-subGaussian in $L_1(T, m)$. Then $\exists T : \ L_1(\Omega, \mathbb{P}) \to L_1(T, m)$ with $\|T\| \leq KC$ (K a numerical constant) such that

$$\forall n \quad T(g_n) = \varphi_n$$

Let $T \in L_1(m_1) \otimes L_1(m_2)$ (algebraic $\otimes$) say $T = \sum x_j \otimes y_j$ then the

projective and injective tensor product norm

denoted respectively by $\| \cdot \|_\wedge$ and $\| \cdot \|_\vee$ are very explicitly described by

$$\|T\|_\wedge = \int |\sum x_j(t_1) y_j(t_2)| dm_1(t_1) dm_2(t_2)$$

$$\|T\|_\vee = \sup\{|\sum \langle x_j, \psi_1\rangle \langle y_j, \psi_2\rangle| \mid \|\psi_1\|_\infty \le 1, \|\psi_2\|_\infty\}.$$

Theorem

Let (φ_n^1) *and* (φ_n^2) $(1 \le n \le N)$ *are subGaussian with constants* C_1, C_2. *Then for any* $0 < \delta < 1$ *there is a decomposition in* $L_1(m_1) \otimes L_1(m_2)$ *of the form*

$$\sum_1^N \varphi_n^1 \otimes \varphi_n^2 = t + r$$

satisfying

$$\|t\|_\wedge \le w(\delta)$$

$$\|r\|_\vee \le \delta,$$

where $w(\delta)$ *depends only on* δ *and* C_1, C_2.
Moreover

$$w(\delta) = O(\log((C_1 C_2)/\delta))$$

Proof reduces to the case $\varphi_n^1 = \varphi_n^2 = g_n$

Proof of Theorem (1)

Let $f = \sum a_n \psi_n^1(t_1)\psi_n^2(t_2)$
$|a_n| = s_n a_n$

Note: (φ_n) *subGaussian* $\Rightarrow (s_n\varphi_n)$ *subGaussian* (same constant)

$S = \sum_1^N s_n \varphi_n^1 \otimes \varphi_n^2 = t + r$

$$\langle f, S\rangle = \sum |a_n|$$

Therefore

$$\sum |a_n| = \langle f, t + r\rangle \leq |\langle f, t\rangle| + |\langle f, r\rangle|$$

$$\leq w(\delta)\|f\|_\infty + \sum |a_n||\langle \psi_n^1 \otimes \psi_n^2, r\rangle| \leq w(\delta)\|f\|_\infty + (\delta C_1' C_2') \sum |a_n|$$

and hence

$$\sum |a_n| \leq (1 - \delta C_1' C_2')^{-1} w(\delta)\|f\|_\infty$$

About Randomly Sidon

Bourgain and Lewko noticed that Slepian's classical comparison Lemma for Gaussian processes implies that randomly $\otimes^k$-Sidon and randomly Sidon are the same property, not implying Sidon. However, we could prove that this implies $\otimes^4$-Sidon:

Theorem (2)

Let (φ_n, ψ_n) be biorthogonal systems both bounded in L_∞. The following are equivalent:

(i) *The system (ψ_n) is randomly Sidon.*

(ii) *The system (ψ_n) is $\otimes^4$-Sidon.*

(iii) *The system (ψ_n) is $\otimes^k$-Sidon for all $k \geq 4$.*

(iv) *The system (ψ_n) is $\otimes^k$-Sidon for some $k \geq 4$.*

This generalizes Rider's result that randomly Sidon implies Sidon for characters

Open question: What about $k = 2$ or $k = 3$?

Non-commutative case

G compact non-commutative group
$\widehat{G}$ the set of distinct irreps, $d_\pi = \dim(H_\pi)$
$\Lambda \subset \widehat{G}$ is called Sidon if $\exists C$ such that for any finitely supported family (a_π) with $a_\pi \in M_{d_\pi}$ $(\pi \in \Lambda)$ we have

$$\sum_{\pi\in\Lambda} d_\pi \mathrm{tr}|a_\pi| \le C\|\sum_{\pi\in\Lambda} d_\pi \mathrm{tr}(\pi a_\pi)\|_\infty.$$

$\Lambda \subset \widehat{G}$ is called randomly Sidon if $\exists C$ such that for any finitely supported family (a_π) with $a_\pi \in M_{d_\pi}$ $(\pi \in \Lambda)$ we have

$$\sum_{\pi\in\Lambda} d_\pi \mathrm{tr}|a_\pi| \le C\mathbb{E}\|\sum_{\pi\in\Lambda} d_\pi \mathrm{tr}(\varepsilon_\pi \pi a_\pi)\|_\infty$$

where (ε_π) are an independent family such that each ε_π is uniformly distributed over $O(d_\pi)$.
Important Remark (easy proof) Equivalent definitions:
• unitary matrices (u_π) uniformly distributed over $U(d_\pi)$
• Gaussian random matrices (g_π) normalized so that $\mathbb{E}\|g_\pi\| \approx 2$ $(\{d_\pi^{1/2} g_\pi \mid \pi \in \Lambda, 1 \le i,j \le d_\pi\}$ forms a standard Gaussian (real or complex) i.i.d. family

Fundamental example

$$G = \prod_{n \geq 1} U(d_n)$$

$$\Lambda = \{\pi_n \mid n \geq 1\}$$

$$\pi_n : \; G \to U(d_n) \quad n\text{-th coordinate}$$

$$C = 1 : \; \sum\nolimits_{n \geq 1} d_n \mathrm{tr}|a_n| = \| \sum\nolimits_{n \geq 1} d_n \mathrm{tr}(\pi_n a_n) \|_\infty .$$

Rider (1975) extended all results previously mentioned to arbitrary compact groups

Note however that the details of his proof that randomly Sidon implies Sidon (solving the non-commutative union problem) never appeared

I plan to remedy this on arxiv soon

General matricial systems

Assume given a sequence of finite dimensions d_n.
For each n let (φ_n) be a random matrix of size $d_n \times d_n$ on (T, m).
We call this a "matricial system".
Let g_n be an independent sequence of random $d_n \times d_n$-matrices, such that $\{{d_n}^{1/2} g_n(i,j) \mid 1 \leq i,j \leq d_n\}$ are i.i.d. normalized $\mathbb{C}$-valued Gaussian random variables. Note $\|g_n(i,j)\|_2 = {d_n}^{-1/2}$.

The **subGaussian condition** becomes: for any N and $y_n \in M_{d_n}$ $(n \leq N)$ we have

$$\| \sum d_n \mathrm{tr}(y_n \varphi_n) \|_{\psi_2} \leq C (\sum d_n \mathrm{tr}|y_n|^2)^{1/2} = \| \sum d_n \mathrm{tr}(y_n g_n) \|_2. \tag{1}$$

In other words, $\{d_n^{1/2} \varphi_n(i,j) \mid n \geq 1, 1 \leq i,j \leq d_n\}$ is a subGaussian system of functions.

The **uniform boundedness condition** becomes

$$\exists C' \ \forall n \quad \|\varphi_n\|_{L_\infty(M_{d_n})} \leq C'. \tag{2}$$

As for the **orthonormality condition** it becomes

$$\int \varphi_n(i,j) \overline{\varphi_{n'}(k,\ell)} = d_n^{-1} \delta_{n,n'} \delta_{i,k} \delta_{j,\ell}. \tag{3}$$

In other words, $\{d_n^{1/2} \varphi_n(i,j) \mid n \geq 1, 1 \leq i,j \leq d_n\}$ is an orthonormal system.

The definition of $\otimes^k$-**Sidon** it now means that the family of *matrix products* $(\varphi_n(t_1)\cdots\varphi_n(t_k))$ is Sidon on $(T,m)^{\otimes^k}$

Theorem (3)

Theorems (1) and (2) are still valid with the corresponding assumptions:

- *subGaussian implies $\otimes^2$-Sidon*
- *randomly subGaussian implies $\otimes^4$-Sidon*

Example of application

Let $\chi \geq 1$ be a constant. Let T_n be the set of $n \times n$-matrices $a = [a_{ij}]$ with $a_{ij} = \pm 1/\sqrt{n}$. Let

$$A_n^\chi = \{a \in T_n \mid \|a\| \leq \chi\}.$$

This set **includes** the famous Hadamard matrices. We have then

Corollary

There is a numerical $\chi \geq 1$ such that for some C we have

$$\forall n \geq 1\ \forall x \in M_n \quad \mathrm{tr}|x| \leq C \sup_{a',a'' \in A_n^\chi} |\mathrm{tr}(xa'a'')|.$$

Equivalently, denoting $A_n^\chi A_n^\chi = \{a'a'' \mid a', a'' \in A_n^\chi\}$ its absolutely convex hull satisfies

$$(\chi)^2 absconv[A_n^\chi A_n^\chi] \subset B_{M_n} \subset C absconv[A_n^\chi A_n^\chi]$$

Curiously, even the case when $|\Lambda| = 1$ (only a single irrep) is interesting
The simplest (and prototypical) example of this situation with $|\Lambda_n| = 1$ is the case when $G_n = U(n)$ the group of unitary $n \times n$-matrices, and Λ_n is the singleton formed of the irreducible representation (in short irrep) defining $U(n)$ as acting on $\mathbb{C}^n$. Sets of this kind and various generalizations were tackled early on by Rider under the name "local lacunary sets"

The next Theorem of course is significant only if $\dim(\pi_n) \to \infty$

Theorem (Characterizing SubGaussian characters)

Let G_n be compact groups, let $\pi_n \in \widehat{G_n}$ be nontrivial irreps, let $\chi_n = \chi_{\pi_n}$ as well as $d_n = d_{\pi_n}$. The following are equivalent.

(i) *$\exists C$ such that the singletons $\{\pi_n\} \subset \widehat{G}_n$ are Sidon with constant C, i.e. we have*

$$\forall n \, \forall a \in M_{d_n} \quad \mathrm{tr}|a| \leq C \sup_{g \in G} |\mathrm{tr}(a\pi_n(g))|.$$

(ii) *$\exists C$ such that $\forall n \quad \|\chi_n\|_{\psi_2} \leq C$.*

(iii) *For each (or some) $0 < \delta < 1$ there is $0 < \theta < 1$ such that*

$$\forall n \quad m_{G_n}\{Re(\chi_n) > \delta d_n\} \leq e\theta^{d_n^2}.$$

(iv) *$\exists C$ such that*

$$\forall n \quad d_n \leq C \int_{U(d_n)} \sup_{g \in G_n} |\mathrm{tr}(u\pi_n(g))| m_{U(d_n)}(du).$$

Although I never had a concrete example, I believed naively for many years that this Theorem could be applied to finite groups. To my surprise, Emmanuel Breuillard showed me that it is not so. By a Theorem of Jordan, any finite group $\Gamma \subset U(d)$ (Breuillard extended this to **amenable** subgroups of $U(d)$) has an Abelian subgroup of index at most $(d+1)!$
This implies for any representation $\pi : \ G \to U(d)$ with **finite range**

$$\int_{U(d)} \sup_{g \in G} |\mathrm{tr}(u\pi(g))| m_{U(d)}(du) \leq c(d \log(d))^{1/2} << d$$

and also

$$\|\chi_\pi\|_{\psi_2} \geq c\sqrt{d/\log(d)} >> 1.$$