Commutants mod Normed Ideals

[from $k_J(\tau)$ to $\mathcal{E}(\tau; J)$]

Dan-Virgil Voiculescu
UC Berkeley

Normed Ideal Perturbations
the Role of the Obstruction to
Quasicentral Approximate Units
relative to the Ideal.

New Object of Interest:
the Commutant mod the
Normed Ideal.

$\mathcal{H}$ complex separable ∞ dim

$(\mathcal{J}, |\ |_{\mathcal{J}})$ normed ideal of compact operators

$(\mathcal{C}_p, |\ |_p)$ p-class, $|T|_p = \left(\sum_j s_j^p\right)^{1/p}$

$(\mathcal{C}_p^-, |\ |_p^-)$ Lorentz $(p,1)$

$$|T|_p^- = \sum_j s_j\, j^{-1+1/p}, \quad (1 \le p \le \infty)$$

$s_1 \geq s_2 \geq \cdots$ eigenvalues of $(T^*T)^{1/2}$

$\tau = (T_1, \dots, T_n)$ n-tuple bdd. operators

$$\mathcal{R}_1^+ = \{ A \mid 0 \le A \le I,\ A \text{ finite rank} \}$$

$$k_{\mathcal{J}}(\tau) = \liminf_{A \in \mathcal{R}_1^+} \max_{1 \le j \le n} |[A, T_j]|_{\mathcal{J}}$$

$$k_{\mathcal{J}}(\tau) = 0 \iff A_n \uparrow I,\ A_n \in \mathcal{R}_1^+,$$
$$|[A_n, T_j]|_{\mathcal{J}} \longrightarrow 0, \quad 1 \le j \le n$$

(quasicentral approximate unit for τ relative to $\mathcal{J}$)

$\mathcal{J} = \mathcal{C}_p \quad k_p(\tau)$, $\mathcal{J} = \mathcal{C}_p^- \quad k_p^-(\tau)$

$k_J(\tau)$ "Size-J dimensional measure of τ"

p-dimensional $\sim$ $J = \mathcal{C}_p^-$

$k_p(\tau) \in \{0, \infty\}$ if $1 < p$ $\quad (\mathcal{C}_1 = \mathcal{C}_1^-)$

τ commuting n-tuple of Hermitian ops.

$$\left(k_n^-(\tau)\right)^n = \gamma_n \int_{\mathbb{R}^n} m(s)\, d\lambda(s)$$

multiplicity function of Lebesgue abs. cont. part of spectral measure

$k_n^-(\tau) = 0 \iff$ spectral measure of τ singular wrt Lebesgue

In general $k_p^-(\tau)$ as function of p decreasing

$0 < k_{p_0}^-(\tau) < \infty \implies$ $k_p^-(\tau) = \infty \quad p < p_0$

$k_p^-(\tau) = 0 \quad p > p_0$

$\tau - \tau' \in J \implies k_J(\tau) = k_J(\tau')$

(6

$p = \infty \qquad k_\infty^-(\tau)$

$$k_\infty^-(\tau) \leq 2\|\tau\| \log(2n-1)$$

$$k_\infty^-(\tau \otimes I_{\mathcal{H}_1}) = k_\infty^-(\tau)$$

$$\mathcal{J} \supset \mathcal{C}_\infty^-, \ \mathcal{J} \neq \mathcal{C}_\infty^- \Rightarrow k_\mathcal{J}(\tau) = 0$$
(all τ)

$S_1, \dots S_n$
creations by $e_1, \dots e_n$
on $\mathcal{T}(\mathbb{C}^n)$
(extended Cuntz)

$$k_\infty^-(S_1, \dots, S_n) = \log n$$

$\bar{k}_\infty$ and entropy

1°. T measure preserving ergodic automorphism of (Ω, Σ, μ), $\mu(\mathcal{X})=1$

U_T induced unitary in L^2

Φ multiplications in L^2 by meas. functions taking finite # of values

$$\mathcal{H}_p(T) = \sup_{\substack{\varphi \subset \Phi \\ \text{finite}}} \bar{k}_\infty(\varphi \cup \{U_T\})$$

$$\mathcal{H}_p(T) \asymp \underset{\text{Kolmogorov–Sinai entropy}}{h(T)}$$

2° μ finitary probability measure on group G with finite generator $g_1, \dots, g_n$

$h(G, \mu) > 0 \Rightarrow k_\infty^-(\lambda(g_1), \dots, \lambda(g_n)) > 0$

Avez entropy of random walk

left regular rep.

Further results on k_∞^- for

Gromov hyperbolic groups

entropy of subshifts

in Rui Okayasu papers

G finitely generated group K generators

$$k_g(\lambda(K)) = \begin{cases} 0 \\ \text{finite} \\ \infty \end{cases}$$ does not depend on choice of K

(generalizes Yamasaki's p-hyper/para-bolicity)

$k_\infty^-(\lambda(K)) = 0 \implies G$ supramenable

(recent result uses Kellerhals-Monod-Rørdam)

Problem:

$$k_\infty^-(\tau) = k_\infty^-(\tau') \overset{?}{\Longrightarrow} k_\infty^-(\tau\otimes\tau') = 0$$

(similar to open question
G, G' supramenable $\overset{?}{\Longrightarrow} G\times G'$ supramenable)

Problem: $k_\infty^-(\lambda(K)) = 0 \overset{?}{\Longleftarrow} G$ supramenable

(i.e. $k_\infty^-(\lambda(K)) = 0 \overset{?}{\Longleftrightarrow} G$ supramenable

Uses of $k_J(\tau)$

Adaptation to normed ideals of the Noncommutative Weyl-v. Neumann Type Theorem

A C^*-alg. with $\underline{1}$, $X_1 \cdots X_n$ generator

ρ_1, ρ_2 $*$-representations on $\mathcal{H}$, $\rho_j(A) \cap \mathcal{K} = \{0\}$, $j = 1,2$

$k_J(\rho_j(\{X_1, \dots, X_n\})) = 0$, $j = 1,2$.

$$\Downarrow$$

$\exists U$ unitary $\quad |U\rho_1(X_k)U^* - \rho_2(X_k)|_J < \varepsilon$

$k = 1, \dots, n$

<u>Cor.</u> N normal $\Rightarrow N = \underset{\text{diagonal}}{D} + \mathcal{C}_2$

$[A = C(\underset{\mathbb{R}^2}{\widehat{K}})$, X_1, X_2 coordinate functions$]$

Generalized singular and absolutely continuous subspaces of τ w.r.t. J

$\mathcal{H} = \mathcal{H}_s(\tau; J) \oplus \mathcal{H}_a(\tau; J)$ τ-reducing

$\mathcal{H}_s(\tau; J)$ largest τ-reducing subspace $\mathcal{X}$ so that $k_J(\tau|\mathcal{X}) = 0$.

(13)

τ n-tuple of commuting Hermitian ops
$\mathcal{J} = \mathcal{C}_n^-$ then:

$\mathcal{H}_s(\tau; \mathcal{C}_n^-) =$ Lebesgue singular subspace $\mathcal{H}_{sing}(\tau)$
$\mathcal{H}_a(\tau; \mathcal{C}_n^-) =$ Lebesgue absolutely cont. $\mathcal{H}_{ac}(\tau)$

$\tau - \tau' \in \mathcal{C}_n^-$ then

$\tau | \mathcal{H}_{ac}(\tau) \underset{\text{unitary}}{\sim} \tau' | \mathcal{H}_{ac}(\tau')$

for $n=1$ consequence of Kato-Rosenblum Thm
for general n proved using $k_n^-(\tau)$ machinery

The Banach algebras $\mathcal{E}(\tau;\mathcal{J})$

$$\tau = \tau^* = (T_j)_{1\le j\le n} \subset \mathcal{B}(\mathcal{H}),\ (\mathcal{J}, |\ |_{\mathcal{J}})$$

$$\mathcal{E}(\tau;\mathcal{J}) = \{X \in \mathcal{B}(\mathcal{H}) \mid [X, T_j] \in \mathcal{J},\ 1 \le j \le n\}$$

$$|||X||| = \|X\| + \max_{1\le j\le n} |[X, T_j]|_{\mathcal{J}}$$

Banach $*$-algebras with isometric involution

$$\mathcal{K}(\tau;\mathcal{J}) = \mathcal{E}(\tau;\mathcal{J}) \cap \mathcal{K}$$

closed 2-sided ideal in $\mathcal{E}(\tau;\mathcal{J})$

$$\mathcal{E}/\mathcal{K}(\tau;\mathcal{J}) = \mathcal{E}(\tau;\mathcal{J}) / \mathcal{K}(\tau;\mathcal{J})$$

If $\mathcal{J} = \mathcal{K}$, $\mathcal{E}/_{\mathcal{K}}(\tau;\mathcal{K}) =$ Paschke dual of $C^*(p(\tau))$
p homomorphism to $\mathcal{B}/\mathcal{K}$

$\mathcal{E}(\tau;\mathcal{J})$ or $\mathcal{E}/_{\mathcal{K}}(\tau;\mathcal{J})$ <u>are not</u> in general somekind of smooth subalgebras of $\mathcal{E}(\tau;\mathcal{K})$ or $\mathcal{E}/_{\mathcal{K}}(\tau;\mathcal{K})$

Much richer K-theory, which reflects perturbation theory facts

τ n-tuple of commuting Hermitian operators

$\sigma(\tau) = [0,1]^n$ to simplify

which implies $K_0(\mathcal{E}(\tau;\mathcal{K})) = 0$.

$\mathrm{mac}(\tau) =$ multiplicity of Lebesgue absolutely continuous part of τ
a.e. defined measurable function $[0,1]^n \longrightarrow \{0,1,2,\dots\infty\}$

$$\mathcal{F}(\tau) = K_0\left(\left(\tau \mid \mathcal{H}_{ac}(\tau)\right)'\right)$$

$$\sim \quad f : [0,1]^n \longrightarrow \mathbb{Z}, \quad f \mid (\mathrm{mac}(\tau))^{-1}(\infty) = 0$$

$$|f(x)| \leq C \ \mathrm{mac}(\tau)(x)$$

a.e. etc.
measurable

1°. $n=1$, $\mathfrak{J}=\mathfrak{C}_1$

$$K_0(\mathcal{E}(T;\mathfrak{C}_1)) \simeq \mathcal{F}(T)$$

$$[P]_0 \rightsquigarrow \mathrm{mac}(P(T\otimes I_n)P)\chi_{\mathrm{mac}(\tau)^{-1}([0,\infty))}$$

2°. $n=1$, $\mathfrak{J}\neq\mathfrak{C}_1$ (means $\mathfrak{J}\supsetneqq\mathfrak{C}_1$)

$$K_0(\mathcal{E}(T;\mathfrak{J}))=0$$

3°. $n\geq 3$, $\mathfrak{J}=\mathfrak{C}_n^-$, assume $\mathcal{H}_{ac}(\tau)=\mathcal{H}$

$$K_0(\mathcal{E}(\tau;\mathfrak{C}_n^-))=\mathcal{F}(\tau)\oplus \underset{\text{unknown}}{X}$$

4°. $n=2$, $\mathfrak{J}=\mathcal{C}_2$

$$K_0(\mathcal{E}(\tau;\mathcal{C}_2)) \longrightarrow L^1_{real}([0,1]^2, d\lambda)$$

$$[P]_0 \rightsquigarrow g_{P(T_1+iT_2)P}$$ Pincus principal function

nontrivial homomorphism
infinite rank group in range

Homomorphisms in 1°, 3°, 4° "canonical":
do not depend on replacing
τ by τ', $\tau \equiv \tau' \bmod \mathfrak{J}$.

Duality, Multipliers, Corona

many similarities between

$\mathcal{K}$, $\mathcal{B}$, $\mathcal{B}/\mathcal{K}$ and

$\mathcal{K}(\tau;\mathcal{J})$, $\mathcal{E}(\tau;\mathcal{J})$, $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$

assume $k_{\mathcal{J}}(\tau)=0$, finite rank dense in $\mathcal{J}$

— $\mathcal{E}/\mathcal{K}(\tau;\mathcal{J})$ is isometrically C^*-subalg. in $\mathcal{B}/\mathcal{K}$

— $\mathcal{E}(\tau;\mathcal{J}) = \mathcal{M}(\mathcal{K}(\tau;\mathcal{J}))$

— if finite rank also dense in $\mathcal{J}^{dual}$
then $\mathcal{E}(\tau;\mathcal{J})$ = bidual of $\mathcal{K}(\tau;\mathcal{J})$.

Countable Degree-1 Saturation

Def. (Farah-Hart)

M C^*-alg., X_m, X_m^*, $m \in \mathbb{N}$ non-commuting indeterminates. Degree 1 $*$-polynomial linear combination of a, aX_mb, aX_m^*b, $a,b \in M$. M is <u>countably degree-1 saturated</u> if for every sequence of degree-1 $*$-polynomials P_n and compact sets K_n, $n \in \mathbb{N}$, TFAE :

(i) there are $b_m \in M$, $m \in \mathbb{N}$ such that $\|b_m\| \leq 1$ and $\|P_n(b)\| \in K_n$ for all $n \in \mathbb{N}$, where $b = (b_1, b_2, \cdots)$

(ii) for every $N \in \mathbb{N}$, there are $b_m \in M$, $\|b_m\| \leq 1$, $m \in \mathbb{N}$ such that $\operatorname{dist}(\|P_n(b)\|, K_n) \leq 1/N$ for all $n \in \mathbb{N}$, $n \leq N$.

<u>Fact</u> Assume $k_J(\tau) = 0$ and finite rank dense in J. Then $\mathcal{E}/\mathcal{K}(\tau; J)$ is countably degree-1 saturated.

(Ref-1

1°. Almost normal operators mod Hilbert-Schmidt and the K-theory of the algebras $E\Lambda(\Omega)$
J. Noncommutative Geom. 8 (2014) no. 4, 1123–1145

2°. (with J. Bourgain) The essential centre of the mod a diagonalization ideal commutant of a n-tuple of commuting hermitian operators
arXiv: 1309.1686

3°. Countable degree-1 saturation of certain C^*-algebras which are coronas of Banach algebras. Groups Geom. Dyn. 8 (2014) 985–1006

Ref-2

4°. Some C^*-algebras which are coronas of non-C^*-Banach algebras, J. of Geometry and Physics 105 (2016), 123–129

5°. A remark about supramenability and the Macaev norm, arXiv: 1605.02135

6°. K-theory and perturbations of absolutely continuous spectra arXiv: 1606.00520