THE MATHEMATICAL LEGACY OF UFFE HAAGERUP

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THEORY OF FILTRATRIONS OF AF-ALGEBRAS AND STANDARDNESS.

"Non-commutative version of the theory of measure-theoretical filtrations".

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We discuss firstly commutative case

1. A filtration is the decreasing sequence of the sigma-fields of measurable sets of the space (X, μ):

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Each sigma field canonically corresponds **to the measurable partition of the space** (X, μ): each elements of sigma-field $\mathfrak{A}$ are the sets which are consist with the blocks of that partition. So filtration uniquely $\mod 0$ generates the decreasing sequence of the measurable partitions $\{\xi_n\}_n$:

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Another type of description of the filtration as decreasing sequence of subalgebras of functions (or, equivalently — sequence of the operators of mathematical expectation on sigma-fields $\mathfrak{A}_i$):

$$L^\infty(X, \mu) \supset L^\infty(X_{\xi_1}, \mu_{\xi_1}) \dots \quad \text{or} \quad Id > P_{\xi_1} > P_{\xi_2} \dots$$

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If we choose a trace χ (central measure on $\mathcal{A}$) then we obtain a filtration of the Past of markov measure with maximal entropy.

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Many problems about filtrations have appeared in ergodic theory, (decreasing sequences of measurable partitions = filtrations of of sigma-fields in the standard measure space), theory of stochastic processes (martingale theory), boundaries (Martin, exit, Poisson-Furstenberg etc.), theory of approximation of the group actions; VWB-Ornstein criteria etc.

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Consider the main partial case, so called *dyadic filtrations*. In measure-theoretic case was studied from the end of 60-th. Dyadic filtration. For AF-algebras:

Definition A filtration $\{\mathcal{A}_n\}$ of AF-algebra $\mathcal{A}$ called dyadic if $\bigcap_n \mathcal{A}_n = \{const\mathbf{1}\}$ and for all $n \in \mathbb{N}$:

$$\mathcal{A} \cong M_{2^n}(\mathbb{C}) \bigotimes \mathcal{A}_n \quad \forall n \in \mathbb{N}, \text{and} \quad \bigcap_n A_n = \{const\} \quad (*)$$

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(Dyadic filtration of AF-algebra (when exists) is an analogue of the notion of countable tensor product of algebra $M_2(\mathbb{C})$)

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3)Action of $\sum Z_2$, Entropy of filtration.
4)Graph of Ordered and Unordered pairs.
(Corresponding Bratteli diagrams, Tower of measures.

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The filtration called "discrete" if the conditional filtration $\{\mathfrak{A}_i/\mathfrak{A}_n; i = 0, 1, \dots n-1\}$ over sigma-field $\mathfrak{A}_n$ for all n are filtration (hierarchy) of the finite space with measure.

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Filtration in the measure space called ergodic, or regular, or Kolmogoroff, or has zero-one-law — if ($\mathfrak{N}$ is trivial sigma-field):

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Definition

Homogeneous standard filtrations of the measure space is a filtration which is isomorphic in measure theoretic sense to Bernoulli filtration (or filtration of product type) with arbitrary components: filtration on the space $\prod_{n-1}^{\infty}(\mathbf{r}_n, m_{r_n})$, where $\mathbf{r}_n$ is finite space with $r_n \in \mathbb{N} \setminus 0$ points, and m_k a uniform measure on $\mathbf{r}_n$. Dyadic filtration: $r_n \equiv 2$

We will give a general definition of standard filtration later.

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A concrete discrete filtration called *Markov filtration* if is the past of a one-sided Markov chain with discrete time, with finite list of transition probabilities and arbitrary state space. Each discrete filtration can be realized as Markov one.

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The group D_n of all automorphisms of n-level dyadic tree with 2^n points acts on each block $C : |C| = 2^n$ of partition $\xi_n, 1, \dots$ and consequently acts on the functions on C.
Let $\eta = \{B_1, B_2, \dots B_k\}$ -an arbitrary finite measurable partition of $[0,1]$ with k blocks. On each block $C \in \xi_n$ define a function $f_n : C \to \mathbf{k} : f_n(c) = i \in \mathbf{k} : c = C \cap B_i$. Denote the orbit of action of the group D_n on the vectors $\{f_n(c)\}_{c \in C}$ as $Orb_n(C)$. Finally define on the set of orbits of the group D_n the metric r_n:
$r_n(O_1, O_2) = \min_{x \in O_1, y \in O_2} \rho_n(x, y)$, where ρ_n is Hamming metric on the vectors with value in $\mathbf{k}$

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Criteria of standardness Dyadic filtration $\{\xi_n\}$ is Bernoulli (or product-type or standard) iff $\forall$ finite measurable partition η

$$\lim_n \int_{[0,1]\times[0,1]} r_n(Orb_n(C), Orb_n(C'))dCdC' = 0.$$

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To describe standard AFC^*-algebras.

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It is make sense in the case of AF-algebras to distinguish weak and strong standardness: weak means that in all II_1 factor representations the image of algebra is standard in the sense which is described below.

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Two filtrations $\{\mathfrak{A}_n\}_{n=0}^{\infty}$ and $\{\mathfrak{A}'_n\}_{n=0}^{\infty}$ called finitely isomorphic if for each N the finite fragments for $n = 0, 1 \ldots N$ of its are metrically isomorphic.(For each n there exists mp automorphism T_n for each $k < n$ $T_n\mathfrak{A}_k = \mathfrak{A}'_k\}$.

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The sets of all conditional measures of almost all elements of the all partitions $\xi_n, n = 1, \dots$ are invariants of the finite isomorphism.
Are there other invariants of ergodic filtrations besides the finite invariants?
The problem of classification of discrete Markov filtrations in the category of measure spaces or in other categories is deep and quite topical.

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Definition

AF-algebra called standard if for any ergodic central measure tail filtration if standard in the previous sense.

Examples of standard graphs: Pascal, Young etc. Concentration, Limit shape theorem.

Definition

The result of transferring the metric ρ_X on the space X to the Borel space Y along the equipped map

$$\phi : X \to Y$$

is the metric ρ_Y on Y defined by the formula

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We define a class of ergodic filtrations — **Standard Filtrations**.
Definition. A filtration $\{\mathfrak{A}_n\}$ of the measure space called standard if is any quotient filtration over partition ξ : $\{\mathfrak{A}_n/\xi\}$ which is finitely isomorphic to it is isomorphic.
This class has the following properties:

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Theorem *So the class of all ergodic filtration is a fibre bundle over set of standard filtrations.*
Standardness is generalization of independence ("eventually independence").

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Theorem(standard criteria). The Markov chain $\{x_n, n \leq 0, x_n \in X_n\}$ (X_n is the state space at moment N and it could be depend on n) the filtration of the past of it called standard if $\forall \epsilon > 0, \exists N \in \mathbb{N} \quad \forall n < -N$and $\quad A_n \subset X_n Prob(A) > 1 - \epsilon$ with the following property

$$E_{x_n, x'_n} \mathrm{Dist}(\mathrm{Prob}(.|x_n), \mathrm{Prob}(.|x'_n)) < \epsilon, x_n, x'_n \in X_n$$

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where *Dist* is a Kantorovich-like metric between conditional measures Prob$(.|x_n)$ as a measures "on the trees of the future" (see below).

Theorem$(V - 71)$ The standard dyadic (more generally, homogeneous) filtration is isomorphic to Bernoulli filtration (=the filtration of the past of the classical Bernoulli scheme).

Comments

The condition in the definition asserts the convergence in probability of the conditional measures in the very strong (unform) metric which take care about hierarchy of the future of the trajectories. This is further strengthen of the martingale theorem which asserts the simple convergence of conditional structures and took place for all ergodic filtrations.

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The convergence in the condition is a strong generalization of weak convergence of empirical (conditional) distributions to the unconditional distribution.
There is no limit distribution but there are very strong concentration of the many dimensional distributions up to coupling which preserve the hierarchy of conditional measures.

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$$\bar{\rho}_{n+1}(x, y) = \bar{k}_{\rho_n}(\mu^{C(x)}, \mu^{C(y)})$$

where $C(x), C(y)$ — elements of ξ_n which contain x, y, and $\bar{k}_\rho$ is revised Kantorovich metric for measures on the tree which was defined before.

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Definition

A filtration $\{\mathfrak{A}_n\}_{n\in\mathbb{N}}$ is called standard if

$$\lim_{n\to\infty} \int\int_{X\times X} \bar{\rho}_n(x,y) d\mu(x) d\mu(y) = 0 \qquad (1)$$

for any initial metric ρ.

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EVANESCENT (or VIRTUALLY) measure metric spaces and Gromov-V. invariants of m-m-spaces.

Theorem on classification of the measure-metric spaces.

$\tau = (X, \mu, \rho)$ admissible metric-measure space.

Consider a map

$$F : (X^{\infty}, \mu^{\infty}) \to M_{\infty}(R_{+}),$$

where

$$F(\{x_n\}_n) = \{\rho(x_i, x_k)\}_{i,k}; n, i, k = 1 \dots$$

Then random matrix

$$F_{*}(\mu^{\infty}) \equiv D\tau$$

("matrix distribution") is the complete invariant of the triple τ w.r. to measure preserving isometry. The map $\tau - -- > D_{\tau}$ is continuous in the right sense.

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("matrix distribution") is the complete invariant of the triple τ w.r. to measure preserving isometry. The map $\tau - -- > D_\tau$ is continuous in the right sense. What happened if there is a sequence of $m - m$ spaces which does not converge?

Virtual matrix distributions.