A survey of weak amenability

Alain Valette

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References:

- U. Haagerup, *An Example of a Non Nuclear C*-Algebra, which has the Metric Approximation Property*, Invent. Math. 50, pp. 279-293 (1978/79).
- M. Cowling and U. Haagerup, *Completely bounded multipliers of the Fourier algebra of a simple Lie group of rank one*, Invent. Math. 96, pp. 507-549 (1989)
-U. Haagerup, *Group C^*-Algebras without the Completely Bounded Approximation Property*, Journal of Lie Theory 26, pp. 861–887 (2016)

1 Definitions

G: a locally compact group, 2nd countable. Recall (Leptin):

G is amenable if and only if there exists a sequence of positive definite, compactly supported functions converging to 1 uniformly on compacts subsets.

This can be generalized in the two following ways:

Definition 1.1. *G has the Haagerup property (or: is a-(T)-menable) if there exists a sequence of continuous positive definite functions on G, vanishing at infinity, converging to 1 uniformly on compact sets.*

Definition 1.2. *G is weakly amenable (or: has the completely bounded approximation property CBAP) if there exists a sequence* $(\phi_n)_{n>0}$ *of continuous, compactly supported functions on G, converging to 1 uniformly on compact sets, with*

$$\sup_n \|\phi_n\|_{M_0 A(G)} < +\infty.$$

Here $\|.\|_{M_0 A(G)}$ is the completely bounded norm on the space $M_0 A(G)$ of completely bounded multipliers of the Fourier algebra $A(G)$. The *Cowling-Haagerup constant* is the best possible Λ with $\sup_n \|\phi_n\|_{M_0 A(G)} \leq \Lambda$.

2 Examples of weakly amenable groups

- amenable groups
- closed subgroups of $SO(n,1)$ (de Cannière-Haagerup 1984) and $SU(n,1)$ (Cowling 1985) (e.g. free groups)
- Coxeter groups, and more generally groups acting properly on finite-dimensional $CAT(0)$ cubical complexes (Guentner-Higson 2010)

In all those cases $\Lambda(G) = 1$. They also have the Haagerup property.

3 More examples

- $G = Sp(n,1)$ $(n \geq 2)$, with $\Lambda(G) = 2n - 1$; and $G = F_{4(-20)}$, with $\Lambda(G) = 21$ (Cowling-Haagerup 1989)
- hyperbolic groups (Ozawa 2007)

Some of those groups do not have the Haagerup property, because they have Kazhdan's property (T).

4 Properties of weak amenability

- For G discrete: can be read off from $C_r^*(G)$ and $vN(G)$.
- For discrete groups: invariant under measure equivalence.
- If Γ is a lattice in G, then $\Lambda(\Gamma) = \Lambda(G)$. (Cowling-Haagerup 1989)

Theorem 4.1. *If N is a closed, amenable subgroup in a weakly amenable group G, then there exists a $N \rtimes G$-invariant state on $L^\infty(N)$ (Ozawa 2010)*

Consequence of last result:

Corollary 4.2. $\mathbf{Z}^2 \rtimes SL_2(\mathbf{Z})$ *is not weakly amenable (Haagerup 1986)*

Proof: Assume by contradiction that $\mathbf{Z}^2 \rtimes SL_2(\mathbf{Z})$ is weakly amenable. Let ϕ be a $\mathbf{Z}^2 \rtimes SL_2(\mathbf{Z})$-invariant state on $\ell^\infty(\mathbf{Z}^2)$. Decompose $\mathbf{Z}^2$ into $SL_2(\mathbf{Z})$-orbits and observe that each non trivial orbit is of the form $SL_2(\mathbf{Z})/A$, with A an abelian subgroup of $SL_2(\mathbf{Z})$. By non-amenability of $SL_2(\mathbf{Z})$, the state ϕ is zero on any non-trivial orbit, so ϕ is evaluation at $(0, 0)$, which is of course not $\mathbf{Z}^2$-invariant. $\square$

Corollary 4.3. $SL_3(\mathbf{R})$ *is not weakly amenable (Haagerup 1986).* $\square$

Compare with Uffe's original proof below.

5 Permanence properties

Property	a-(T)-men	Weak amenability	Remarks
Closed subgroups	Yes	$\Lambda(H) \leq \Lambda(G)$	
Products	Yes	$\Lambda(G_1 \times G_2) = \Lambda(G_1)\Lambda(G_2)$	
Free products	Yes	$\Lambda(G_1 * G_2) = 1$	Only for $\Lambda = 1$
	Jolissaint 2001	Ricard-Xu 2006	OPEN IN GENER
Graph products	Antolin-P	Reckwerdt 2015	SAME
	Dreesen 2013		

6 Cowling's question

Based on experimental evidence, around 1998, M. Cowling conjectured that: *the class of Haagerup groups coincides with the class of CMAP groups, i.e.* $\Lambda = 1$.
Disproved in 2007:

Theorem 6.1. *1. Haagerup property is stable under wreath products (Cornulier-Stalder-V)*

2. If $\Lambda \neq \{1\}$ *and* Γ *is non-amenable, then* $\Lambda \wr \Gamma$ *is not weakly amenable (Ozawa-Popa).*

So $C_2 \wr \mathbf{F}_2$ is an example of a not weakly amenable group with the Haagerup property.
However: Cowling's conjecture holds for interesting subclasses:

- closed subgroups of $SO(n,1)$ and $SU(n,1)$;
- groups acting properly on finite-dimensional $CAT(0)$ cubical complexes (Guentner-Higson).

A finitely generated group G is a *generalized Baumslag-Solitar of rank* n if it admits a co-compact action on some tree, such that vertex and edge stabilizers are isomorphic to $\mathbf{Z}^n$. Such a group admits a canonical homomorphism $hol : G \to GL_n(\mathbf{R})$, the *holonomy representation*.

Theorem 6.2. *(Cornulier-V, 2013) For a generalized Baumslag-Solitar group G of rank n, TFAE:*

1. *$\overline{hol(G)}$ is amenable;*
2. *G has the Haagerup property;*
3. *G is weakly amenable.*

In that case: $\Lambda(G) = 1$.

Observations:

- There is no known direct connection between the two properties. In all cases, it is an *a posteriori* observation that a given class of groups satisfy both properties.
- It seems that the discrepancy is related to some lack of finiteness condition.

Conjecture 1. *For groups admitting a finite-dimensional* $\underline{EG}$ *(= classifying space of proper actions): Haagerup property is equivalent to weak amenability with* $\Lambda = 1$

7 A proof of weak amenability

Theorem 7.1. *(R. Szwarc, 1991) Let G be a group acting properly on a locally finite tree T. Then $\Lambda(G) = 1$.*

Examples: free groups, $SL_2(\mathbf{Q}_p)$,...

Main steps in the proof:

- (Bozejko, Fendler, Gilbert) Let $\phi : G \to \mathbf{C}$ be a continuous function. If there exists continuous, bounded functions $u, v : G \to \mathcal{H}$ such that $\phi(y^{-1}x) = \langle u(x)|v(y)\rangle$ then $\phi \in M_0A(G)$. In particular, for π a uniformly bounded representation of G, coefficients of π belong to $M_0A(G)$.

- (Pytlik-Szwarc 1986, Szwarc 1991, V. 1996) Let D be the open unit disk in $\mathbf{C}$. Fix a base-vertex $v_0 \in T$. There exists an analytic family $(\pi_z)_{z\in D}$ of uniformly bounded representations of G on $\ell^2(T)$ such that:

1. $\langle \pi_z(g)\delta_{v_0} | \delta_{v_0} \rangle = z^{d(gv_0,v_0)}$;
2. π_0 is the permutation representation, and π_t is unitary for $t \in]-1,1[$;
3. $\sup_{g\in G} \|\pi_z(g)\| \leq \frac{2|1-z^2|}{1-|z|}$.

- Let γ_r be the circle of radius r in D, set $\pi_{\gamma_r} = \int_{\gamma_r}^{\oplus} \pi_z \, |dz|$. Then, for any function f holomorphic on a neighborhood of γ_r, the function $\phi(g) = \int_{\gamma_r} z^{d(gv_0,v_0)} f(g) \, dz$ is a coefficient of π_{γ_r}.

- For $n \in \mathbf{N}$, take $f(z) = \frac{z^{-(n+1)}}{2\pi i}$. Then $\chi_n(g) = \frac{1}{2\pi i} \int_{\gamma_r} z^{d(gv_0,v_0)} z^{-(n+1)} \, dz$ is the characteristic function of the sphere $\{g \in G : d(gv_0, v_0) = n\}$. Optimizing over r, get $\|\chi_n\|_{M_0A(G)} \leq \frac{e}{2}(n+1)$

- Set $\phi_t(g) = t^{d(gv_0,v_0)}$. For $t \to 1$, it converges to 1 uniformly on compact sets, and $\|\phi_t\|_{M_0A(G)} = 1$ as ϕ_t is positive-definite. BUT: not compactly supported! To fix that, set $\phi_{t,n} = \sum_{k=0}^{n} t^k \chi_k$. Then

$$\|\phi_t - \phi_{t,n}\|_{M_0A(G)} = \|\sum_{k=n+1}^{\infty} t^k \chi_k\|_{M_0A(G)} \leq \frac{e}{2} \sum_{k=n+1}^{\infty} f^k (k+1)$$

  that goes to 0 for $n \to \infty$.

8 Recent developments

A weaker notion than weak amenability was introduced by Haagerup and Kraus (1994):
A locally compact group G has the *Approximation Property (AP)* if there is a sequence $(\phi_n)_{n>0}$ in $A(G)$ such that $\phi_n \to 1$ in the $\sigma(M_0A(G), M_0(A(G))_*)$-topology, where $M_0(A(G))_*$ denotes the natural predual of $M_0A(G)$.
Haagerup and Kraus:

- If Γ is a lattice in G, Γ has AP if and only if G has AP.
- (unlike Haagerup property and weak amenability) AP is stable under extensions (so: $\mathbf{Z}^2 \rtimes SL_2(\mathbf{Z})$ has AP).

Establishing a conjecture of Haagerup and Kraus: a simple Lie group of rank at least 2, does not have AP (Lafforgue and de la Salle 2011, Haagerup and de Laat 2013 and 2016).