Structure and classification of free Araki-Woods factors

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Shlyakhtenko's free Araki-Woods factors

- Orthogonal representation $(U_t)_{t\in\mathbb{R}}$
  $\rightsquigarrow$ von Neumann algebra (M, φ) with faithful normal state.
- Direct sum $(U_t \oplus V_t)_{t\in\mathbb{R}} \rightsquigarrow$ free product $(M, \varphi) * (N, \psi)$.
- Intertwiner T between U and V with $\|T\| \leq 1$
  $\rightsquigarrow$ state preserving completely positive $\theta : (M, \varphi) \to (N, \psi)$.
- A free probability analog of the CAR,
  generalizing Voiculescu's free Gaussian functor.
- **Open problem:**
  classify these von Neumann algebras M in terms of $(U_t)_{t\in\mathbb{R}}$.

Construction: full Fock space

- Given a Hilbert space H, construct the full Fock space

$$\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}$$

- For $\xi \in H$, left creation operator $\ell(\xi) : \mathcal{F}(H) \to \mathcal{F}(H)$ given by

$$\ell(\xi)\Omega = \xi \quad \text{and} \quad \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$

- Vacuum state $\varphi(T) = \langle T\Omega, \Omega \rangle$.

Theorem (Voiculescu, 1983)

- The operator $s(\xi) = \ell(\xi) + \ell(\xi)^*$ has Wigner's semicircular distribution with radius $2\|\xi\|$ w.r.t. φ.
- If $\xi \perp \eta$, then $s(\xi)$ and $s(\eta)$ are $*$-free w.r.t. φ.
- For $H = \mathbb{C}^n$, we have $L(\mathbb{F}_n) \cong \{\ell(e_i) + \ell(e_i)^* \mid i = 1, \ldots, n\}''$.

Construction: free Araki-Woods factors

Let H be a Hilbert space and $K_{\mathbb{R}} \subset H$ a real subspace satisfying

- $K_{\mathbb{R}} \cap \mathrm{i}K_{\mathbb{R}} = \{0\}$
- $K_{\mathbb{R}} + \mathrm{i}K_{\mathbb{R}} \subset H$ is dense.

Definition (Shlyakhtenko, 1996)

Define $\Gamma(K_{\mathbb{R}} \subset H)'' = \{\ell(\xi) + \ell(\xi)^* \mid \xi \in K_{\mathbb{R}}\}''$ acting on $\mathcal{F}(H)$.

The vacuum state $\varphi(T) = \langle T\Omega, \Omega \rangle$ is faithful and called the **free quasi-free state.**

Basic question: classify $\Gamma(K_{\mathbb{R}} \subset H)''$ in terms of $K_{\mathbb{R}} \subset H$;

An equivalent point of view

Let $(U_t)_{t\in\mathbb{R}}$ be an orthogonal representation on the real Hilbert space $H_\mathbb{R}$.

- Put $H = H_\mathbb{R} + \mathrm{i} H_\mathbb{R}$
  and $J : H \to H : J(\xi + \mathrm{i}\eta) = \xi - \mathrm{i}\eta$ for all $\xi, \eta \in H_\mathbb{R}$.
- Define Δ on H such that $\Delta^{\mathrm{i}t} = U_t$.
- Put $S = J\Delta^{1/2}$ and $K_\mathbb{R} = \{\xi \in D(S) \mid S(\xi) = \xi\}$.
- Then, $K_\mathbb{R} \cap \mathrm{i}K_\mathbb{R} = \{0\}$ and $K_\mathbb{R} + \mathrm{i}K_\mathbb{R} \subset H$ is dense.

⤳ Every such $K_\mathbb{R} \subset H$ arises in this way.

⤳ Write $\Gamma(U, H_\mathbb{R})'' = \Gamma(K_\mathbb{R} \subset H)'' = \{\ell(\xi) + \ell(S(\xi))^* \mid \xi \in D(S)\}''$.

Note: conversely $S(\xi + \mathrm{i}\eta) = \xi - \mathrm{i}\eta$ for all $\xi, \eta \in K_\mathbb{R}$ and then $S = J\Delta^{1/2}$.

Connes' invariants for free Araki-Woods factors

Write $M = \Gamma(U, H_{\mathbb{R}})''$ with free quasi-free state φ.

Generators: $s(\xi) = \ell(\xi) + \ell(S(\xi))^*$ with $\sigma_t^{\varphi}(s(\xi)) = s(U_t\xi)$.

Theorem (Shlyakhtenko, 1996-1998)

Unless $H_{\mathbb{R}} = \mathbb{R}$ and $U_t = \mathrm{id}$, we have that M is a factor

- of type II_1 iff $U_t = \mathrm{id}$ for all $t \in \mathbb{R}$,
- of type III_λ iff U is periodic with period $2\pi/|\log\lambda|$,
- of type III_1 iff U is not periodic,
- that is full: $\mathrm{Inn}(M) \subset \mathrm{Aut}(M)$ is closed,
- with Connes' τ-invariant, i.e. the topology on $\mathbb{R}$ induced by $\mathbb{R} \to \mathrm{Out}(M) : t \mapsto \sigma_t^{\varphi}$, equal to the topology induced by $t \mapsto U_t$,
- that is almost periodic iff U is almost periodic, in which case

$\mathrm{Sd}(M) = \mathrm{Sd}(U) :=$ subgroup of $\mathbb{R}_+^*$ generated by the eigenvalues of U.

Almost periodic free Araki-Woods factors

A full factor M is called **almost periodic** if it admits a faithful normal state φ such that $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ is almost periodic.

Then, $\mathrm{Sd}(M) \subset \mathbb{R}_+^*$ is defined such that the compactification given by $t \mapsto \sigma_t^\varphi \in \mathrm{Out}(M)$ corresponds to $\mathbb{R} \subset \widehat{\mathrm{Sd}(M)}$.

Theorem (Shlyakhtenko, 1996)

The almost periodic free Araki-Woods factors M are fully classified by their Sd invariant $\mathrm{Sd}(M) \subset \mathbb{R}_+^*$.

So, for almost periodic orthogonal representations U and V, we have $\Gamma(U)'' \cong \Gamma(V)''$ if and only if $\mathrm{Sd}(U) = \mathrm{Sd}(V)$.

Attention: only the "non trivial" case, because $\Gamma(\mathrm{id}, H_\mathbb{R})'' \cong L(\mathbb{F}_{\dim(H_\mathbb{R})})$.

Isomorphisms through Shlyakhtenko's matrix models.

Note: unique free Araki-Woods factor of type III_λ, $\lambda \in (0,1)$.

Beyond the almost periodic case

Until now: no new **quantitative** invariants for free Araki-Woods factors.

But: a number of **qualitative** results, mostly based on the Connes-Takesaki **continuous core** $\mathrm{core}(M) = M \rtimes_{\varphi} \mathbb{R}$.

Write $M = \Gamma(U, H_{\mathbb{R}})''$.

- Shlyakhtenko (1997). When U is a multiple of the regular representation, then $\mathrm{core}(M) \cong L(\mathbb{F}_{\infty}) \overline{\otimes} B(K)$.

  When all tensor powers $U_t \otimes \cdots \otimes U_t$ are disjoint from the regular representation, then $\mathrm{core}(M) \ncong L(\mathbb{F}_t) \overline{\otimes} B(K)$.
- Shlyakhtenko (2002): two **non isomorphic** free Araki-Woods factors having the same τ invariant.
- Houdayer (2008): when U is mixing, then $\mathrm{core}(M)$ is solid.
- Hayes (2015): when U is disjoint from the regular representation, then $\mathrm{core}(M) \ncong L(\mathbb{F}_t) \overline{\otimes} B(K)$.

Classification of orthogonal representations

Given a Borel measure μ on $\mathbb{R}$ that is symmetric, i.e. $\mu(X) = \mu(-X)$,

put $H_{\mathbb{R}} = \{\xi \in L^2(\mathbb{R}, \mu) \mid \xi(-x) = \overline{\xi(x)}\}$ with $(U_t\xi)(x) = \exp(\mathrm{i}tx)\xi(x)$.

- Every orthogonal representation of $\mathbb{R}$ is orthogonally isomorphic with a direct sum of such $(U, H_{\mathbb{R}})$.
- Orthogonal representations of $\mathbb{R}$ are thus fully classified by

  a symmetric measure μ on $\mathbb{R}$ and

  a symmetric **multiplicity function** $m : \mathbb{R} \to \mathbb{N} \cup \{+\infty\}$

  (that we always assume to satisfy $m(x) \geq 1$ for μ a.e. x)

- Two such (μ_i, m_i) define the same rep iff $\mu_1 \sim \mu_2$ and $m_1 = m_2$ a.e.

⤳ We write $\Gamma(\mu, m)''$ for the free Araki-Woods factor.

Note: the spectral measure of $U \otimes V$ is $\mu_U * \mu_V$.

Note: almost periodic = atomic measure μ.

Non almost periodic free Araki-Woods factors

Consider the set $\mathcal{S}(\mathbb{R})$ of symmetric probability measures μ on $\mathbb{R}$ such that

- writing $\mu = \mu_c + \mu_a$,
- we have $\mu_c * \mu_c \prec \mu_c$,
- μ_a is not concentrated on $\{0\}$.

Write $\Lambda(\mu_a) =$ subgroup of $\mathbb{R}$ generated by the atoms of μ_a.

Theorem (Houdayer–Shlyakhtenko–V, 2016)

For $\mu \in \mathcal{S}(\mathbb{R})$, the free Araki-Woods factors $\Gamma(\mu, m)''$ are exactly classified by the subgroup $\Lambda(\mu_a) \subset \mathbb{R}$ and the measure class of $\mu_c * \delta_{\Lambda(\mu_a)}$.

Here: δ_Λ is any atomic probability measure with set of atoms Λ.

Source of many examples:

Start with μ_0 and a non trivial μ_a. Take $\mu = \mu_a \vee \bigvee_{n \geq 0} \mu_0^{*n}$.

In particular: many non isomorphic $\Gamma(\mu, m)''$ with the same τ invariant.

States with non amenable centralizer

Recall: $M^\psi = \{x \in M \mid \forall y \in M : \psi(xy) = \psi(yx)\}$.

Theorem (Houdayer–Shlyakhtenko–V, 2016)

Let $M = \Gamma(\mu, m)''$ be a free Araki-Woods factor with free quasi-free state φ.

If ψ is any faithful normal state on M such that M^ψ is non amenable, then

- there exist non zero projections $p \in M^\varphi$ and $q \in M^\psi$,
- and a partial isometry $v \in M$ with $v^*v = p$ and $vv^* = q$, such that

$\psi(x) = \lambda\, \varphi(v^*xv)$ for all $x \in qMq$, with $\lambda = \psi(q)/\varphi(p)$.

Main consequence:

if $\Gamma(\mu, m)'' \cong \Gamma(\nu, n)''$ and if $\mu(t) > 0$ for some $t \neq 0$,

there also exists an isomorphism preserving the free quasi-free states.

And then the measure class of $\bigvee_{n\geq 1} \mu^{*n}$ becomes an invariant.

The bicentralizer problem

- **Connes' question:** does every III_1 factor have a trivial bicentralizer ?
- **Haagerup:** yes for the hyperfinite III_1 factor !
- **Haagerup's reformulation:** trivial bicentralizer

  iff there exists a faithful normal state ψ such that $(M^\psi)' \cap M = \mathbb{C}1$,

  iff the set of such ψ is dense among all normal states on M.

⟶ Often, M^ψ is a II_1 factor. **But:**

Theorem (Houdayer–Shlyakhtenko–V, 2016)

Let $M = \Gamma(\mu, m)''$ with μ continuous. For every faithful normal state ψ on M, we have that M^ψ is amenable.

Houdayer (2008): free Araki-Woods factors have a trivial bicentralizer.

Dependence on the multiplicity function

To start with: $\Gamma(\delta_0, m)'' \cong L(\mathbb{F}_{m(0)})$.

- Let λ be the Lebesgue measure. Then, $\Gamma(\lambda + \delta_0, 1)'' \not\cong \Gamma(\lambda + \delta_0, 2)''$.

  Reason: one has all centralizers amenable and the other not.

- All $\Gamma((\lambda, +\infty) + (\delta_0, m))''$ with $2 \leq m < +\infty$ are isomorphic,

  but whether they are isomorphic with $m = +\infty$ is equivalent with the question $L(\mathbb{F}_m) \cong L(\mathbb{F}_\infty)$.

Intriguing open cases:

- Does $\Gamma(\lambda|_{[-a,a]}, m)''$ depend on $a > 0$ and/or $m \in \mathbb{N}$?
- Are $\Gamma(\lambda, 1)''$ and $\Gamma(\lambda + \delta_0, 1)''$ isomorphic ?

  Both have all centralizers amenable and core $L(\mathbb{F}_\infty) \mathbin{\overline{\otimes}} B(K)$.

  The free quasi-free state has trivial centralizer, resp. diffuse abelian centralizer.

Deformation/rigidity and the conjugacy of states

Let φ and ψ be faithful normal states on a von Neumann algebra M.

We say that a corner of φ is conjugate to a corner of ψ if

- there exist non zero projections $p \in M^\varphi$ and $q \in M^\psi$,
- and a partial isometry $v \in M$ with $v^*v = p$ and $vv^* = q$, such that

$\psi(x) = \lambda\, \varphi(v^*xv)$ for all $x \in qMq$, with $\lambda = \psi(q)/\varphi(p)$.

- Two realizations of core(M): as $M \rtimes_\varphi \mathbb{R}$ and as $M \rtimes_\psi \mathbb{R}$.
- In this way, $L_\varphi(\mathbb{R}) \subset \text{core}(M)$ and $L_\psi(\mathbb{R}) \subset \text{core}(M)$.

Theorem (Houdayer–Shlyakhtenko–V, 2016)

A corner of φ is conjugate to a corner of ψ if and only if $L_\varphi(\mathbb{R}) \prec L_\psi(\mathbb{R})$ inside core(M) in the sense of Popa's intertwining-by-bimodules.

Further applications: free products

Let μ be a continuous symmetric probability measure.

Define $M = \Gamma(\mu, +\infty)''$ with its free quasi-free state φ.

Theorem (Houdayer–Shlyakhtenko–V, 2016)

If (A, τ) and (B, τ) are nonamenable II_1 factors with their trace, then $(M, \varphi) * (A, \tau)$ is isomorphic with $(M, \varphi) * (B, \tau)$ if and only if there exists $t > 0$ such that $A \cong B^t$.

Note: isomorphisms are not assumed to be state preserving.

But again: up to corners and ..., there then exists a state preserving isomorphism.

Further applications: many free products of amenable von Neumann algebras are **not** isomorphic to free Araki-Woods factors.

Strong solidity

Free Araki-Woods factors really are "type III free group factors".

Free group factors $M = L(\mathbb{F}_n)$

- (Voiculescu, 1995) have no Cartan subalgebra,
- (Ozawa, 2003) are solid: $A' \cap M$ is amenable whenever $A \subset M$ diffuse,
- (Ozawa–Popa, 2007) are strongly solid: $\mathcal{N}_M(A)''$ is amenable whenever $A \subset M$ is diffuse and amenable.

Free Araki-Woods factors $M = \Gamma(\mu, m)''$

- (Shlyakhtenko, 2003) are solid,
- (Houdayer–Ricard, 2010) have no Cartan subalgebra.

Note: only consider subalgebras that are the range of a faithful normal conditional expectation.

Strong solidity for free Araki-Woods factors

Theorem (Boutonnet–Houdayer–V, 2015)

All free Araki-Woods factors are strongly solid.

- Let $M = \Gamma(\mu, m)''$ be a free Araki-Woods factor with its free quasi-free state φ.
- Finite corners $p \operatorname{core}(M) p$ of the continuous core fall under the Ozawa-Popa theorem: tracial von Neumann algebras with Haagerup's CMAP and good deformation properties.
- But: the normalizer of $A \subset M$ induces a **generalized** (groupoid/pseudogroup type) normalizer of $\operatorname{core}(A)$ inside $\operatorname{core}(M)$.
- Extend the Ozawa-Popa theorem to cover as well these generalized normalizers:

  we prove that tracial von Neumann algebras with CMAP and a malleable deformation in the sense of Popa are **stably strongly solid.**