

Linearization and Brown measure

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Section 1

Uffe's Legacy in Free Probability Theory



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On Voiculescu's R - and S -Transforms for Free Non-Commuting Random Variables

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1 Introduction

Voiculescu introduced in 1986-87 (cf. [8], [9]) the R - and S -transforms for the distribution of "random variables" in a non-commutative algebra $\mathcal{A}$ with a fixed probability measure (i.e., a linear functional $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ for which $\varphi(1) = 1$), and proved the addition formula for the R -transform

$$R_{\mu_{a+b}}(z) = R_{\mu_a}(z) + R_{\mu_b}(z), \quad (1.1)$$

Brown's Spectral Distribution Measure for R -Diagonal Elements in Finite von Neumann Algebras

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In 1983 L. G. Brown introduced a spectral distribution measure for non-normal elements in a finite von Neumann algebra $\mathcal{M}$ with respect to a fixed normal faithful tracial state τ . In this paper we compute Brown's spectral distribution measure in case T has a polar decomposition $T = UH$ where U is a Haar unitary and U and H are $*$ -free. (When $\text{Ker } T = \{0\}$ this is equivalent to that (T, T^*) is an R -diagonal pair in the sense of Nica and Speicher.) The measure μ_T is expressed explicitly in terms of the S -transform of the distribution μ_{T^*T} of the positive operator T^*T . In case T is a circular element, i.e., $T = (X_1 + iX_2)\sqrt{2}$ where (X_1, X_2) is a free semicircular system, then $\text{sp } T = D$, the closed unit disk, and μ_T has constant density $1/\pi$ on D . © 2000 Academic Press

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A new application of random matrices: $\text{Ext}(C_{\text{red}}^*(F_2))$ is not a group

By UFFE HAAGERUP and STEEN THORBJØRNSSEN*

Dedicated to the memory of Gert Kjærgård Pedersen

Abstract

In the process of developing the theory of free probability and free entropy, Voiculescu introduced in 1991 a random matrix model for a free semicircular system. Since then, random matrices have played a key role in von Neumann algebra theory (cf. [V8], [V9]). The main result of this paper is the following extension of Voiculescu's random matrix result: Let $(X_1^{(n)}, \dots, X_r^{(n)})$ be a system of r stochastically independent $n \times n$ Gaussian self-adjoint random matrices as in Voiculescu's random matrix paper [V4], and let $(x_1, \dots, x_r)$ be

Brown measure

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invariant subspaces

- with H. Schultz
- with K. Dykema

Linearization

Haagerup and
Thorbjørnsen



operator algebraic applications

- $Ext(C_{red}^*(FF_2))$ not group with S. Thorbjørnsen
- no projections in $C^*(F_2)$ with S. Thorbjørnsen, H. Schultz

Brown measure

Haagerup and Larsen



new ideas and concepts for dealing with limits of random matrices and functions of free random variables

- Brown measure as candidate for asymptotic eigenvalue distribution of non-Hermitian random matrices
- limits of norms of polynomials in independent random matrices
- calculation of distribution of functions of free variables

Linearization

Haagerup and Thorbjørnsen



Section 2

Linearization



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Voiculescu, Kirchberg, Pisier, etc.

non-linear problem $\rightarrow$ operator-valued linear problem

$$p(x_1, \dots, x_m) \rightarrow \alpha_0 \otimes 1 + \alpha_1 \otimes x_1 + \dots + \alpha_m \otimes x_m$$

Example

Is xy invertible? We have

$$\begin{pmatrix} xy & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & x \\ y & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

Hence xy is invertible if and only if

$$\begin{pmatrix} 0 & x \\ y & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes y$$

is invertible

Theorem (Haagerup, Thorbjørnsen (+Schultz); Anderson)

Every polynomial $p(x_1, \dots, x_m)$ has a (non-unique) linearization

$\hat{p} = \alpha_0 \otimes 1 + \alpha_1 \otimes x_1 + \dots + \alpha_m \otimes x_m$ such that

$$(z - p)^{-1} = [(\Lambda(z) - \hat{p})^{-1}]_{1,1}, \quad \text{where} \quad \Lambda(z) = \begin{pmatrix} z & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

and hence also

$$G_p(z) = \varphi((z - p)^{-1}) = [\varphi \otimes 1(\Lambda(z) - \hat{p})^{-1}]_{1,1} = [G_{\hat{p}}(\Lambda(z))]_{1,1}.$$

- $G_p(z) = \varphi[(z - p)^{-1}]$ is the Cauchy transform of p
- $G_{\hat{p}}(b) = \varphi \otimes 1[(b - \hat{p})^{-1}]$ is the operator-valued Cauchy transform of $\hat{p}$

Historical remark

Note that, as we only became aware in recent years, this linearization trick is also a well-known idea in many other mathematical communities, known under various names like

- Higman's trick (Higman "The units of group rings", 1940)
- recognizable power series (automata theory, Kleene, Schützenberger)
- linearization by enlargement (ring theory, Cohn)
- system realization (control theory, Helton, Vinnikov)

So maybe Uffe's trick is not so new, but his applications in operator algebras, free probability and random matrices are nevertheless amazing ...

Theorem (Haagerup, Thorbjørnsen)

For $X_1^{(N)}, \dots, X_r^{(N)}$ independent Gaussian random matrices, and $s_1, \dots, s_r$ free semicircular elements we have for any polynomial p in r non-commuting variables almost surely

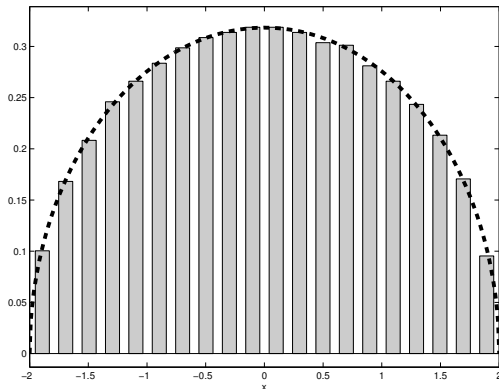
$$\lim_{N \rightarrow \infty} \|p(X_1^{(N)}, \dots, X_r^{(N)})\| = \|p(s_1, \dots, s_r)\|$$

Generalizations

This has been generalized to many other situations

- real and symplectic case (Schultz)
- other types of random matrices
 - ▶ Wigner or Wishart matrices (Capitaine, Donati-Martin; Anderson)
 - ▶ Haar unitary random matrices (Collins, Male)
 - ▶ including deterministic matrices (Male)
- “exact separation of eigenvalues” (Haagerup, Schultz, Thorbjørnsen)
- non-commutative rational functions instead of polynomials (Yin)

One-matrix case: classical random matrix case



$$s = l + l^*,$$

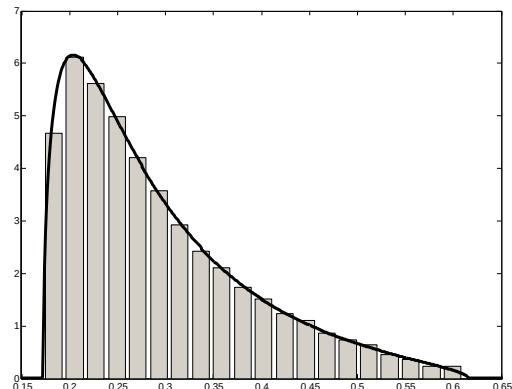
l one-sided shift on
 $\bigoplus_{n \geq 0} \mathbb{C} e_n$

$$\varphi(a) = \langle a e_0, e_0 \rangle$$

$$\mu = \varphi \circ E_s$$

- $X^{(N)} \rightarrow s$ in distribution (Wigner 1955)
- $\|X^{(n)}\| \rightarrow \|s\| = 2$ (Füredi, Komlós 1981)

Multi-matrix case: non-commutative case



$$s_1 = l_1 + l_1^*, s_2 = l_2 + l_2^*$$

$$\bullet r(X_1^{(N)}, X_2^{(N)}) \rightarrow r(s_1, s_2)$$

in distribution (Voiculescu)

$$\bullet \|r(X_1^{(N)}, X_2^{(N)})\|$$

$$\rightarrow \|r(s_1, s_2)\| \text{ (HT)}$$

$$(1 - x_1)^{-1} + (1 - x_1)^{-1} x_2 ((1 - x_1) - x_2 (1 - x_1)^{-1} x_2)^{-1} x_2 (1 - x_1)^{-1}$$

$$= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 - x_1 & -x_2 \\ -x_2 & 1 - x_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Question

Can we actually calculate

- the distribution/norm of $r(s_1, s_2)$

Solution: again by linearization (Belinschi, Mai, Speicher; Helton)

- we need the distribution of $r(s_1, s_2)$
- this is determined by the operator-valued distribution of its linearization

$$\hat{r}(s_1, s_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 - s_1 & -s_2 \\ 0 & -s_2 & 1 - s_1 \end{pmatrix}$$

Question

Can we actually calculate

- the distribution/norm of $r(s_1, s_2)$

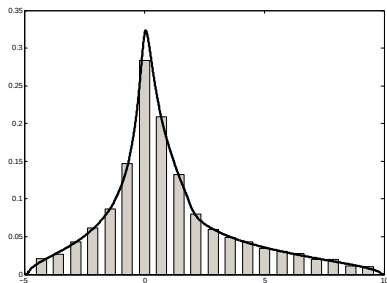
Solution: again by linearization (Belinschi, Mai, Speicher; Helton)

- we need the distribution of $r(s_1, s_2)$
- this is determined by the operator-valued distribution of its linearization
- but this is now an additive (operator-valued) problem

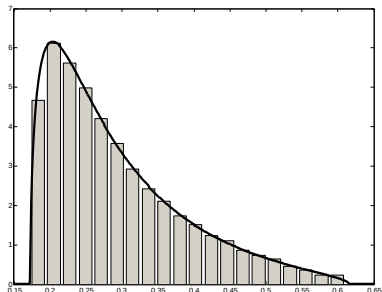
$$\hat{r}(s_1, s_2) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes 1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes s_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \otimes s_2$$

- for this we have analytic theory of operator-valued free convolution

This allows to calculate (by also involving some numerical solving of fixed point equations for the Cauchy transforms) the solid lines in the pictures



$$p(x_1, x_2) = x_1x_2 + x_2x_1 + x_1^2$$



$$r(x_1, x_2)$$

Relevance of p being selfadjoint

Note that whether p is selfadjoint or not

- is not an issue for the norm, as

$$\|p\|^2 = \|pp^*\|$$

→ the Haagerup-Thorbjørnsen Theorem is for arbitrary polynomials

- is an issue for convergence of distribution, if we want to understand the latter as a probability measure (like: eigenvalue distribution for matrices)

→ theory of Belinschi-Mai-Speicher is only for selfadjoint polynomials

Section 3

Brown Measure



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Idea of Brown measure μ_a for operator a

Normal case

Let $x \in (\mathcal{A}, \varphi)$ be normal operator (e.g., selfadjoint or unitary). Then there exists uniquely determined probability measure μ_x on $\mathbb{C}$ such that

$$\varphi(x^n x^{*m}) = \int_{\mathbb{C}} z^n \bar{z}^m d\mu_x(z) \quad \forall m, n \geq 0$$

General, not necessarily normal, case

Let $a \in (\mathcal{A}, \varphi)$ be general operator. Its distribution in algebraic sense is given by collection of all $*$ -moments

$$\varphi(a^{i_1} \cdots a^{i_n}), \quad n \in \mathbb{N}, i_1, \dots, i_n \in \{1, *\}$$

This information cannot be encoded by a probability measure on $\mathbb{C}$, but still we would like to have such a μ_a which captures as much information about all $*$ -moments of a as possible.

Rough idea

In principle, we want to define the Brown measure μ_a by the requirement that μ_a and a have the same Cauchy transform

$$\int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu_a(z) = \varphi\left(\frac{1}{\lambda - a}\right).$$

- For normal a it suffices to have this outside the spectrum of a , for $\lambda \notin \sigma(a)$; there both sides make sense and are analytic functions in λ
- For non-normal a one needs also the information inside the spectrum; the left-hand side makes then still sense almost surely (as a non-analytic function); but the right hand side is problematic

Rough idea

- Instead of

$$\int_{\mathbb{C}} \frac{1}{\lambda - z} d\mu_a(z) = \varphi\left(\frac{1}{\lambda - a}\right)$$

we consider the integrated version

$$\int_{\mathbb{C}} \ln |\lambda - z| d\mu_a(z) = \varphi(\ln |\lambda - a|) = \log \Delta(|a - \lambda|)$$

where Δ is the Fuglede-Kadison determinant (which makes sense in a tracial von Neumann algebra setting)

- The latter formulation makes sense for any $\lambda \in \mathbb{C}$ and defines μ_a uniquely.

History

- L. Brown 1986: Lidskii's theorem in the type II case
- Haagerup, Larsen 2000

Impact on operator algebraic, free probability and random matrix side

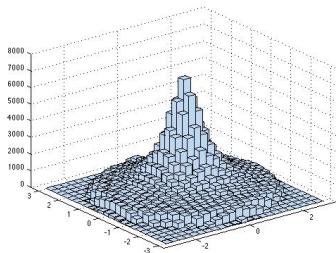
- calculation of Brown measures for known and also new classes of operators
 - ▶ R -diagonal operators and variations (Haagerup, Larsen; Haagerup, Schultz; Biane, Lehner)
 - ▶ DT operators (Dykema, Haagerup)
- investigations on invariant subspaces (Dykema, Haagerup; Haagerup, Schultz)
- Brown measure gives candidate for limit of eigenvalue distribution of non-normal random matrices
 - ▶ single ring theorem (Guionnet, Krishnapur, Zeitouni)

Limits of non-normal random matrices

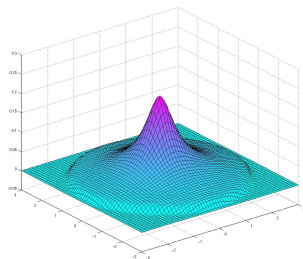
Conjecture

Let $X_1^{(N)}, \dots, X_r^{(N)}$ be r independent Gaussian $N \times N$ random matrices and $s_1, \dots, s_r$ be r free semicirculars. Then, for any polynomial in r non-commuting variables we have that the Brown measure (i.e., the eigenvalue distribution) of $p(X_1^{(N)}, \dots, X_r^{(N)})$ converges, almost surely, weakly to the Brown measure of $p(s_1, \dots, s_r)$.

Example ($p(x_1, x_2, x_3) = x_1x_2x_3 + 2x_2x_3x_1 + x_3x_1x_2$)



?



Section 4

Brown Measure and Linearization

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Hermitization method

Idea: transform non-normal problem into operator-valued selfadjoint problem

We have

$$\int_{\mathbb{C}} \log |\lambda - z| d\mu_a(z) = \log \Delta(|a - \lambda|) = \int_0^\infty \log(t) d\mu_{|a - \lambda|}(t).$$

Hence in order to calculate the Brown measure of a

- we need to calculate the distribution of all hermitian operators $|a - \lambda|$,
- which can be gotten from the operator-valued distribution of

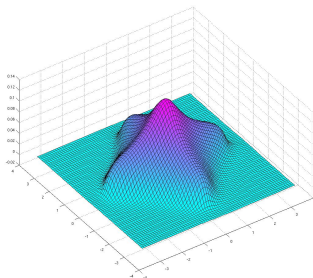
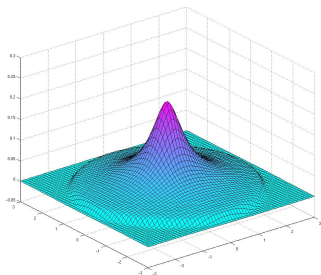
$$A = \begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}$$

Hermitization method

History

- folklore in random matrix theory (going back to Girko?) for eigenvalue distribution
- contact with free probability and operator-valued description on formal level by physicists
 - ▶ Zee, Feinberg
 - ▶ Janik, Nowak, Papp, Zahed
- first rigorous calculations using operator-valued description by
 - ▶ Aagaard, Haagerup (for quasi-nilpotent DT -operator)
- general rigorous theory by
 - ▶ Belinschi, Sniady, Speicher

This allows then the calculation of the Brown measure of any polynomial (or even non-commutative rational function) in free variables



Thanks, Uffe, for the wonderful mathematics!