MEANING UNITARY OPERATORS

Expressing a linear operator T on a complex Hilbert space $\mathcal{H}$ as a convex combination of unitary operators:

$$T = a_1 U_1 + \cdots + a_n U_n \,,\quad a_j \geq 0 \,,\quad \Sigma a_j = 1$$

($U_j \in \mathfrak{A}$ if $T \in \mathfrak{A}$, where $\mathfrak{A}$ is a C^*-algebra).

Murray-von Neumann, R.O. II, TAMS 1937

A is self-adjoint, $\|A\| \leq 1$ (equivalently, $\mathrm{sp} A \subseteq [-1,1]$):

$A = \frac{1}{2}(U + U^*)$, where $U = A + i\sqrt{I - A^2}$, $U^* = A - i\sqrt{I - A^2}$

U and U^* are unitary (and in $\mathfrak{A}$ if $A \in \mathfrak{A}$)

BACKGROUND

$\mathcal{H}$ complex Hilbert space, $\langle x, y \rangle$ inner product of x and y
$\|x\| = \langle x, x\rangle^{1/2}$. T an operator on $\mathcal{H}$ (linear transformation of $\mathcal{H}$ into $\mathcal{H}$) is continuous iff $\sup\{\|Tx\| : x \in \mathcal{H}, \|x\| \leq 1\}$ $(= \|T\|) < \infty$ (T is bounded).
$\mathcal{B}(\mathcal{H})$ = all bounded operators on $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ is an algebra - usual addition and multiplication.
$T \to \|T\|$ is a norm on $\mathcal{B}(\mathcal{H})$. $\mathcal{B}(\mathcal{H})$ is a Banach algebra.

Associated topology on $\mathcal{B}(\mathcal{H})$ is the <u>norm</u> topology.
A C^*-algebra $\mathfrak{A}$ is a norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ such that $T^* \in \mathfrak{A}$ when $T \in \mathfrak{A}$. We assume $I \in \mathfrak{A}$ ($Ix = x$ all x in $\mathcal{H}$).

With T in $\mathfrak{A}$, $T = \frac{1}{2}(T+T^*) + i\frac{1}{2i}(T-T^*)$. If $\|T\| \le 1$, $\|\frac{1}{2}(T+T^*)\| \le 1$ and $\|\frac{1}{2i}(T-T^*)\| \le 1$. So $\frac{1}{2}(T+T^*) = \frac{1}{2}(U_1 + U_1^*)$ and $\frac{1}{2i}(T-T^*) = \frac{1}{2}(U_2 + U_2^*)$. $T = \frac{1}{2}(U_1 + U_1^* + iU_2 + iU_2^*)$.
$\|\Sigma a_j U_j\| \le \Sigma a_j \|U_j\| = \Sigma a_j = 1$. Convex combinations lie in $(\mathfrak{A})_1$, the unit ball of $\mathfrak{A}$ ($\{T \in \mathfrak{A} : \|T\| \le 1\}$).
(1951 Annals) Extreme points of $(\mathfrak{A})_1$ are described. Those of $(\mathcal{B}(\mathcal{H}))_1$ are V such that either or both of $V^*V = I$, $VV^* = I$ holds. (V is an isometry or the adjoint of an isometry.)
If V is a non-unitary isometry, then V is not a convex combination of unitary operators.
$\mathcal{U}(\mathfrak{A})$ is the set of unitary operators in $\mathfrak{A}$.
$\mathrm{co}\,\mathcal{U}(\mathfrak{A})$ is the convex hull of $\mathcal{U}(\mathfrak{A})$, $(\mathrm{co}\,\mathcal{U}(\mathfrak{A}))^=$ is the norm closure of $\mathrm{co}\,\mathcal{U}(\mathfrak{A})$.

Russo - Dye Theorem. $co\,\mathcal{U}(\mathfrak{A})$ is norm dense in $(\mathfrak{A})_1$ $\quad ((co\,\mathcal{U}(\mathfrak{A}))^{=} = (\mathfrak{A})_1)$.

(G.K. Pedersen 1985) If $S \in \mathfrak{A}$ and $\|S\| < 1 - \frac{2}{n}$ for some integer $n\,(>2)$, then there are $U_1, \cdots, U_n$ in $\mathcal{U}(\mathfrak{A})$ such that $S = \frac{1}{n}(U_1 + \cdots + U_n)$.

Each element of the open unit ball $(\mathfrak{A})_1^{\circ}$ $(= \{T \in \mathfrak{A} : \|T\| < 1\})$ is a <u>mean</u> of unitary operators in $\mathfrak{A}$ - there is even an estimate of the number needed: n if S lies more than $\frac{2}{n}$ below the surface. With $n = 3$, if $S \in \mathfrak{A}$ and $\|S\| < 1 - \frac{2}{3} = \frac{1}{3}$, then S is a mean of 3 unitary operators in $\mathfrak{A}$.

<u>Proof of Theorem</u>. With T in $(\mathfrak{A})_1^{\circ}$ and V in $\mathcal{U}(\mathfrak{A})$, $(V+T)/2 = V(I + V^*T)/2$ and $\|V^*T\| = \|T\| < 1$. Thus $I + V^*T$, and hence $(V+T)/2$ are invertible. $((I+V^*T)^{-1} = I - (V^*T) + (V^*T)^2 - \cdots)$
So $(V+T)/2 = UH$ with U in $\mathcal{U}(\mathfrak{A})$ and $H \geq 0$ in $\mathfrak{A}$. Now $\|H\| = \|UH\| \leq 1$, whence $H = \frac{1}{2}(U_1 + U_2)$ with $U_1 + U_2$ in $\mathcal{U}(\mathfrak{A})$. Thus $V + T = UU_1 + UU_2$, with UU_j in $\mathcal{U}(\mathfrak{A})$. (Gardner 1984).

$$V+(n-1)T = U_1+V_1+(n-2)T = U_1+U_2+V_2+(n-3)T = \cdots$$

$$(*) \qquad = U_1+\cdots+U_{n-2}+V_{n-2}+T = U_1+\cdots+U_{n-2}+U_{n-1}+U_n,$$

with U_j in $\mathfrak{A}$ and U_j unitary. If S in $\mathfrak{A}$ is such that $\|S\|<1-\frac{2}{n}$ and $n\geq 3$, then

$$\|(n-1)^{-1}(nS-I)\| \leq (n-1)^{-1}(n\|S\|+1) < 1.$$

Replacing T by $(n-1)^{-1}(nS-I)$ and V by I in $(*)$:

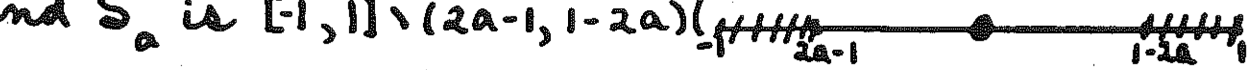
$$nS = U_1+\cdots+U_n \qquad (U_j \in \mathcal{U}(\mathfrak{A})) \quad \blacksquare$$

How good is the estimate n; after all $H=\frac{1}{2}(U_1+U_2)$? With V a non-unitary isometry and $1-\frac{2}{n-1}<a_n<1-\frac{2}{n}$, a_nV has norm a_n and is a mean of n unitary operators but no fewer than n.

Why the insistence on means? If T is a convex combination of n unitary operators in $\mathfrak{A}$, how mean can we get?

THEOREM Very! If $T=a_1U_1+\cdots+a_nU_n$ with U_j in $\mathcal{U}(\mathfrak{A})$, then $T=b_1V_1+\cdots+b_nV_n$ for each $(b_1,\ldots,b_n)$ in $\mathrm{co}\{a_{\pi(1)},\ldots,a_{\pi(n)} : \pi\in Z_n\}$ $(=\mathcal{K})$ and some V_j in $\mathcal{U}(\mathfrak{A})$. In particular, $T=\frac{1}{n}(W_1+\cdots+W_n)$ for some W_j in $\mathcal{U}(\mathfrak{A})$.

The proof involves a detailed combinatorial-geometric analysis of the "permutation polytope" $\mathcal{K}$ and the following "asymmetric" extension of the Murray-von Neumann decomposition.

THEOREM If A is self-adjoint (in $\mathfrak{A}$), $0 \le a \le \frac{1}{2}$, and S_a is $[-1,1] \setminus (2a-1, 1-2a)$ (

) then $A = aU_1 + (1-a)U_2$ with U_1, U_2 in $\mathcal{U}(\mathfrak{A})$ iff $\operatorname{sp} A \subseteq S_a$.

$$u_{\mathfrak{A}}(T) = \min\{n : T = \sum_{j=1}^{n} a_j U_j \;,\; a_j \ge 0, \sum a_j = 1, U_j \in \mathcal{U}(\mathfrak{A})\}$$

$u_{\mathfrak{A}}(T)$ is the <u>unitary rank</u> of T (relative to $\mathfrak{A}$).

$u_{\mathfrak{A}}(V) = \infty$ when V is a non-unitary isometry in $\mathfrak{A}$.

$u_{\mathfrak{A}}(a_n V) = n$ when $1 - \frac{2}{n-1} < a_n < 1 - \frac{2}{n}$.

$u_{\mathfrak{A}}(A) = 2$ when $A = A^*$ and $\|A\| \le 1$.

The unitary rank $u(\mathfrak{A})$ of $\mathfrak{A}$ is $\max\{u(T) : T \in (\mathfrak{A})_1\}$.

$u(\mathcal{R}) = 2$ if $\mathcal{R}$ is a finite von Neumann algebra (hence, for $\mathcal{R}$ the algebra of $n \times n$ complex matrices).

$u(\mathcal{R}) = \infty$ for other von Neumann algebras (e.g. $\mathcal{B}(\mathcal{H})$).

Are there $\mathfrak{A}$ for which $u(\mathfrak{A})$ is not 2 or ∞?

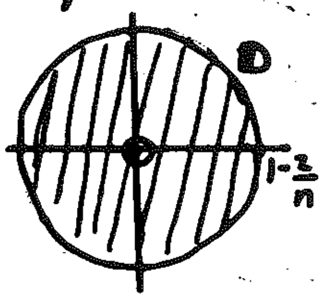
THEOREM (M. Rørdam, Annals '88) If the invertible elements of $\mathfrak{A}$ are norm dense in $\mathfrak{A}$, then $u(\mathfrak{A})$ is 1, 2, or 3; if not, then $u(\mathfrak{A}) = \infty$ and there are elements in $(\mathfrak{A})_1$ at distance 1 from the invertibles; these are precisely those elements that are not in co $\mathcal{U}(\mathfrak{A})$.

What if $T \in \mathfrak{A}$ and $\|T\| = 1 - \frac{2}{n}$?

(Olsen-Pedersen JFA '86) If $\mathfrak{A}$ is a von Neumann algebra $u_{\mathfrak{A}}(T) \leq n$.

$u_{\mathfrak{A}}(T) \leq n$ if $n = 3$ or 4 ($\mathfrak{A}$ a C^*-algebra).

If T is normal ($T^*T = TT^*$), $u_{\mathfrak{A}}(T) \leq n$ (Haagerup '86)



If $\operatorname{sp} T \neq \mathbb{D}$, then $u_{\mathfrak{A}}(T) \leq n$.

THEOREM (Haagerup) $u_{\mathfrak{A}}(T) \leq n$, when $\|T\| = 1 - \frac{2}{n}$.

<u>Uses</u>: (1) Refined norm estimates - example:

$\varphi : \mathfrak{A} \to \mathfrak{B}$; $\varphi(A) \geq 0$ when $A \geq 0$, $\|\varphi(I)\| \leq 1 \Rightarrow \|\varphi\| \leq 1$.

(2) Gelfand-Neumark conjecture (1943) $*$ is isometric.

(3) Non-commutative topology (??)-relation to the invertibles.