Monod V+VI

Theorem Let G act isometrically on L^1 ($L^1([0,1])$, $L^1(\mathbb{R})$, $\ell^1(\mathbb{N})$). If G preserves some bounded set, then $\exists$ F.P. (not necessarily in the set)

$G \curvearrowright \ell^1(G) \supseteq \mathrm{Prob}(G)$,

Derivations: $f \mapsto f'$

Definition A derivation $\mathcal{D}$ of an algebra A is a linear map $A \to A$ s.t.
$\mathcal{D}(ab) = a\mathcal{D}(b) + \mathcal{D}(a)b$.

More generally: A-Bimodule B, $\mathcal{D}: A \to B$,
$C^1 \to C^0$, $f \mapsto f'$.

Question (Williamson 1963).
Are there non-trivial derivations on $L^1(G)$? (into itself).
Trivial means $\exists b \in B\ \forall a \in A: \mathcal{D}(a) = ab - ba$.

Theorem (Haagerup 1983)
Let A be a C^*-algebra. Then every derivation $A \to A^*$ is trivial

Derivations $\longleftrightarrow$ affine actions

$\mathcal{D}: A \longrightarrow B$, $G = A^\times$ (multiplicative group of units).
affine action on B: $g \cdot b = gb + \mathcal{D}(g)g^{-1}$,

$\beta(g) := \mathcal{D}(g)g^{-1} \Longrightarrow \beta(gh) = g \cdot \beta(h) + \beta(g)$.
$\mathcal{D}$ is trivial $\Longleftrightarrow$ the action has a FP.
[In order ~~or~~ to go from affine actions to derivations one can extend derivation on $A^\times$ to A as every element can be written as a sum of 4 (or 3) unitaries.

What spaces do we consider?
$\ell^1 G$, $L^1(G)$, $M(X)$ — bounded measures on locally compact space X, A^* for A a C^*-algebra.

$M(X) = C_0(X)^*$. We look at Banach spaces V s.t. V^* happens to be von Neumann algebra.

All these are examples of
Defn V is an L-space if $V^{**} \cong V_\bullet \oplus_{\ell^1} V_0$, an isometric identification.

KØBENHAVNS UNIVERSITET
UNIVERSITY OF COPENHAGEN

WWW.KU.DK

Always, $V \hookrightarrow V^{**}$, $v \mapsto$ Evaluation at v.

<u>Yoshida–Hewitt</u> (1952)

$L^1(\mathbb{R})$ is an L-space. ~~(L^1)~~ $L^1 \subseteq (L^\infty)^* = L^1 \oplus_{\ell^1} V_0$

$\mu : L^\infty \longrightarrow \mathbb{R}$

$\mathbb{1}_E \longmapsto \mu(E)$

This is additive but not necessarily σ-additive.

In fact $L^\infty(\mathbb{R})^* = \{$finitely additive measures on $\mathbb{R}\}$.

~~Takesi~~

<u>Takesaki</u> (1958) If A is C^*, then A^* is an L-space.

More generally the predual of a von Neumann algebra is an L-space.

<u>Proof of fixed point theorem for L^1</u>

(And any L-space V).

$G \curvearrowright V$ by isometries, preserves $0 \neq K \subseteq V$ bounded

<u>Idea</u>

Consider $C_V(K) = \bigcap$ of $\bigcap$ of closed balls

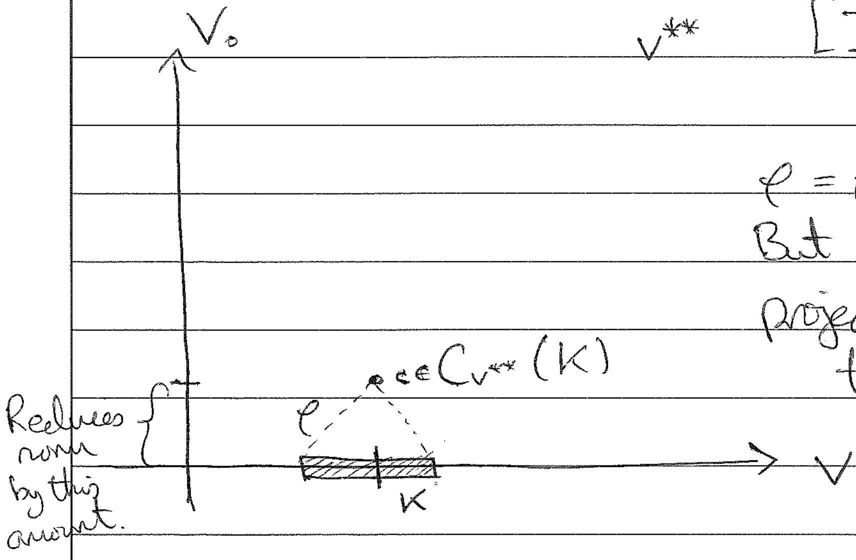
Is $C_V(K)$ empty?

Work in $V^{**} \supseteq V$, $C_{V^{**}}(K) \neq \emptyset$ and compact for weak-$*$ topology.

But the weak-$*$ topology on $(V^*)^*$ is the weak topology when restricted to V.
If $C_{V^{**}}(K) \subseteq V$ then it is weakly compact.
Then RN would provide a G-FP in $C_{V^{**}}(K)$.

<u>Claim</u>

$C_{V^{**}}(K) = C_V(K) \subseteq V$ ~~[If you are on~~] [If V is an L-space].



$\rho = \rho_{V^{**}}(K)$
But one can take a projection of c onto K and take a smaller ρ.
So $\rho = \rho_{V^{**}}(K) = \rho_V(K)$.

<u>Upshot</u> $\forall K \subseteq L^1$ bounded, $C(K) \neq \emptyset$ and weakly compact.

G amenable $\Leftrightarrow$ FP for all convex compact $K \subseteq V$ LCTVS.

<u>Assume V separable</u>

<u>Klee (1955)</u> $K \Rightarrow K' \subset \ell^2$ (norm)

affine homeomorphism

KØBENHAVNS UNIVERSITET
UNIVERSITY OF COPENHAGEN

WWW.KU.DK

(*) What is the class of G s.t. $\forall G$-affine action on ℓ^2 with bounded orbit, $\exists$ F.P ?

Amenability $\Rightarrow$ (*).

<u>FACT</u> $G = \mathbb{N}_0! =$ bijection$(\mathbb{N})$ has (*).

<u>Theorem</u>

If G is l.c. σ-compact then (*) $\Leftrightarrow$ amenability

i.e. G not amenable $\Rightarrow \exists$ fixed point free action.