2016.05.05

Lubotzky I + II

1) Random simplicial complexes
2) Overlapping properties
3) Property testing
4) Quantum error correcting codes

X = f. simplicial cx., i.e. X is a set of subsets of a vertex-set V which is closed under inclusion, i.e. for $F \in X$ and $G \subset F$, $G \in X$.

"Face" $F \in X$ of card. i has dim. $i-1$. Let $X(i) = \{F \in X \mid \dim(F) = i\}$; in part. $X(-1) = \{\emptyset\}$. Define

$$d = \dim(X) = \max\{\dim(F) \mid F \in X\}$$

and say that X is pure if every maximal face is of dim. d.

$$C_i = C_i(X, \mathbb{F}_2) = \mathbb{F}_2\text{-v.s. gen. by } X(i)$$

$$C^i = C^i(X, \mathbb{F}_2) = \{f : X(i) \longrightarrow \mathbb{F}_2\}$$

$$= \text{power set of } X(i).$$

Define $d = d_i : C_i \to C_{i-1}$ by

$$d(F) = \sum_{\substack{G \subset F \\ \dim(G) = \dim(F) - 1}} G$$

and $d = d^i : C^i \to C^{i+1}$ by

$$d(1_F) = \sum_{\substack{F \subset G \\ \dim(G) = \dim(F) + 1}} 1_G$$

Identifying $C_i \to C^i$ by $F \mapsto 1_F$, define

$$\langle f, g \rangle = \sum_{F \in X(i)} f(F) g(F).$$

Prop (1) $d_{i-1} d_i = 0$

(2) $d^{i+1} d^i = 0$

(3) If $f \in C^i$ and $g \in C^{i-1}$, then

$$\langle dg, f \rangle = \langle g, df \rangle.$$

Define homology and cohomology by

$$H_i(X, \mathbb{F}_2) = \ker(d_i) / \operatorname{im}(d_{i-1})$$

$$H^i(X, \mathbb{F}_2) = \ker(d^i) / \operatorname{im}(d^{i+1}).$$

Prop $\dim(H_i) = \dim(H^i)$.

Prop For a general $f \in C^i$,

$$d^i(f)(G) = f(d_{i+1} G).$$

Ex $X = (V, E)$ graph.

$$\begin{array}{ccc} C^0 & \xrightarrow{d^0} & C^1 \\ \wr\uparrow 1 & & \\ \mathcal{P}(V) & & \end{array}$$

$$d^0(1_Y)(e) = 1_Y(d_1 e) = 1_Y(e_+) + 1_Y(e_-)$$

$$= E(Y, X \setminus Y).$$

So $Y \in Z^0$ if and only if Y is a union of connected components:

$$Z^0 = \{ f : V \to \mathbb{F}_2 \mid f \text{ locally const.} \}$$

$$B^0 = \{ f : V \to \mathbb{F}_2 \mid f \text{ constant} \}$$

Hence,

$$\dim H^0(X, \mathbb{F}_2) = \#(\pi_0(X)) - 1.$$

Exercise: Prove directly that $\dim H_0(X, \mathbb{F}_2) = \#(\pi_0(X)) - 1$.

<u>Cor</u> If $V \neq \emptyset$, then X is connected if and only if $H^0(X, \mathbb{F}_2) = 0$.

<u>Def</u> (coboundary expansion à la L-M) Let X be a simplicial complex of pure dimension d. For $0 \leq i < d$,

$$\tilde{\varepsilon}_i(X) = \min \left\{ f \in C^i \setminus B^i \;\middle|\; \frac{|d^i(f)|}{|[f]|} \right.$$

where $[f] = f + B^i$. //

(<u>Exercise</u>: $|[f]| = \min \{ |g| \mid g \in [f] \}$
$= \operatorname{dist}(f, B^i)$.)

calc. this is an NP complete problem; with $\mathbb{R}$-coeff., this is easy linear alg. //

<u>Ex</u> $X = (V, E)$ graph.

$$\tilde{\varepsilon}_0 = \min \left\{ f \in C^0 \setminus B^0 \;\middle|\; \frac{|d(f)|}{|[f]|} \right\}$$

$$= \min_{\emptyset \neq Y \subsetneq V} \left\{ \frac{|E(Y, V \setminus Y)|}{\min(|Y|, |V \setminus Y|)} \right.$$

$[Y] = \{ 1_Y + 0, 1_Y + 1_V \} = 1_{V \setminus Y}$

$|[Y]| = \min(|Y|, |V \setminus Y|)$

Observation : For a finite simplicial complex X,

$$\tilde{\varepsilon}_i(X) > 0 \iff H^i(X, \mathbb{F}_2) = 0.$$

Rmk In the random graph with vertex set $V = [n]$ with an edge between two vertices with probability, the treshold probability for the graph to be connected is

$$p \leq (1-\varepsilon)\frac{\log n}{n}.$$

More generally, a theorem of Meshulam - Walloch states that the treshold probability for the vanishing of

$$H^{d-1}(\text{random } X)$$

is $d \cdot \log n / n$. So up to this treshold, X is not a high dim. expander; and above the treshold, it is.

Def For $F \in X(i)$, let

$$c(F) = \#\{G \in X(d) \mid F \subset G\}$$

and define the weight of F to be

$$w(F) = c(F) \Big/ \binom{d+1}{i+1} |X(d)| .$$

Now, for $f \in C^i$, set

$$\|f\| = \sum_{F \in \mathrm{supp}(f)} w(F)$$

and define

$$\varepsilon_i(X) = \min \left\{ \frac{\|d^i f\|}{\|[f]\|} \;\middle|\; f \in C^i \setminus B^i \right\} .$$

Thm. (L-M, Gromov) Let $X = \Delta_n^{(d)}$ be the complete d-dim. simpl. complex on n points. For all $0 \leq i < d$,

$$\varepsilon_i(X) \geq 1 - o_n(1) .$$

⌈ Def A general X is called an ε-coboundary expander, if for all $0 \leq i < d$, $\varepsilon_i(X) \geq \varepsilon$. ⌋

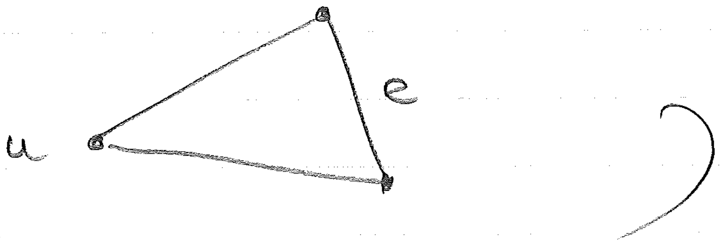
Pf for $d=2$ $\quad \varepsilon_1(X) = ?$

Let $\alpha \in C^1(X)$. For $e \in X(1)$, $u \in X(0)$, $u \notin e$, define $\alpha_u \in C^0(X)$ by

$$\alpha_u(v) = \begin{cases} \alpha((u,v)) & v \neq u \\ 0 & v = u \end{cases}$$

Lemma $\quad (\alpha - d^0(\alpha_u))(e) = \begin{cases} d^1(\alpha)(ue) & u \notin e \\ 0 & u \in e \end{cases}$

(Here $ue = u * e$ is the unique



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Pf of lemma: Check all possibilities. /

Continue pf. of thm.:

$$3\,|d^1(\alpha)| = |\{(u,T) \in X(0)\times X(2) \mid u \in T \in d^1(\alpha)\}|$$

$$\overset{\text{lemma}}{=} |\{(u,e) \mid e \in \alpha - d^0(\alpha_u)\}|$$

$$= \sum_{u \in X(0)} |\alpha - d^0(\alpha_u)|$$

$$\geq n\,|[\alpha]|.$$

This shows that

$$\frac{|d'(\alpha)|}{|[\alpha]|} \geq \frac{w}{3}$$

Renormalizing, this shows

$$\frac{\| d'(\alpha) \|}{\| [\alpha] \|} \geq 1$$

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