Complex semisimple quantum groups

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Locally compact quantum groups

Definition

A Hopf C^*-algebra is a C^*-algebra H together with a nondegenerate $*$-homomorphism $\Delta : H \to M(H \otimes H)$ satisfying the coassociativity relation

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$$

and the density conditions

$$[\Delta(H)(H \otimes 1)] = H \otimes H = [\Delta(H)(1 \otimes H)].$$

Here $[X]$ denotes the closed linear span of a subset X of a Banach space.

Definition (Kustermans-Vaes)

A locally compact quantum group G is given by a Hopf C^*-algebra $H = C_0(G)$ together with faithful left invariant Haar weight ϕ and a faithful right invariant Haar weight ψ on $C_0(G)$.

Locally compact quantum groups

The following basic examples motivate the terminology.

Example

a) Let G be a locally compact group. Then the algebra $H = C_0(G)$ of continuous functions on G vanishing at infinity becomes a locally compact quantum group with the comultiplication

$$\Delta : C_0(G) \to M(C_0(G) \otimes C_0(G)) = C_b(G \times G)$$

given by

$$\Delta(f)(s,t) = f(st).$$

b) Let G be a locally compact group and let $H = C^*_{\text{red}}(G)$ be the reduced group C^*-algebra of G. Then H is a locally compact quantum group with comultiplication

$$\hat{\Delta} : C^*_{\text{red}}(G) \to M(C^*_{\text{red}}(G) \otimes C^*_{\text{red}}(G))$$

given by

$$\hat{\Delta}(u_s) = u_s \otimes u_s.$$

Locally compact quantum groups

Let G be a locally compact quantum group, and let $\mathcal{H} = L^2(G)$ be a GNS-construction for the left Haar weight ϕ of G.

We consider

$$\mathcal{N}_\phi = \{f \in C_0(G) \mid \phi(f^*f) < \infty\}$$

and let $\Lambda : \mathcal{N}_\phi \to \mathcal{H}$ be the GNS-map.

It is a nontrivial fact that one obtains a unitary operator W on $\mathcal{H} \otimes \mathcal{H}$ by defining

$$W^*(\Lambda(f) \otimes \Lambda(g)) = (\Lambda \otimes \Lambda)(\Delta(g)(f \otimes 1))$$

for $f, g \in \mathcal{N}_\phi$.

As in the case of finite groups, the unitary W is multiplicative, which means that it satisfies the pentagon equation

$$W_{12} W_{13} W_{23} = W_{23} W_{12}.$$

Locally compact quantum groups

The GNS-representation of $C_0(G)$ on $\mathcal{H}$ is faithful, so that one can view $C_0(G)$ as a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$.

One can describe the Hopf C^*-algebra structure of $C_0(G)$ completely in terms of W by

$$C_0(G) = [(\mathrm{id} \otimes \mathcal{L}(\mathcal{H})_*)(W)], \qquad \Delta(f) = W^*(1 \otimes f)W.$$

Moreover, one obtains a second Hopf C^*-algebra $C^*_{\text{red}}(G)$ by setting

$$C^*_{\text{red}}(G) = [(\mathcal{L}(\mathcal{H})_* \otimes \mathrm{id})(W)], \qquad \hat{\Delta}(x) = \hat{W}^*(1 \otimes x)\hat{W}.$$

This generalizes the construction of the reduced group C^*-algebra of a locally compact group.

It can be shown that $W \in M(C_0(G) \otimes C^*_{\text{red}}(G)) \subset \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$.

The quantum group $SU_q(2)$

Let $q \in (0, 1]$.

By definition, the *algebra of polynomial functions on* $SU_q(2)$ is the universal $*$-algebra $\mathcal{O}(SU_q(2))$ generated by elements α and γ satisfying the relations

$$\alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha, \quad \gamma\gamma^* = \gamma^*\gamma,$$
$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1.$$

These relations are equivalent to saying that

$$u = \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$$

is a unitary matrix.

The quantum group $SU_q(2)$

The comultiplication $\Delta : \mathcal{O}(SU_q(2)) \to \mathcal{O}(SU_q(2)) \otimes \mathcal{O}(SU_q(2))$ is defined by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \qquad \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma$$

on the generators.

Equivalently, we can write

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j$$

for the entries of the matrix u.

The counit and antipode for $\mathcal{O}(SU_q(2))$ are given by

$$\epsilon(u^i_j) = \delta_{ij}, \qquad S(u^i_j) = (u^j_i)^*$$

respectively.

This turns $\mathcal{O}(SU_q(2))$ into a Hopf $*$-algebra.

The quantum group $SU_q(2)$

Consider the enveloping C^*-algebra $C(SU_q(2))$ of $\mathcal{O}(SU_q(2))$. Then Δ extends to a unital $*$-homomorphism

$$\Delta : C(SU_q(2)) \to C(SU_q(2)) \otimes C(SU_q(2))$$

which turns $C(SU_q(2))$ into a compact quantum group.

In the case $q = 1$ this construction yields the algebra of continuous functions on $SU(2)$ with its comultiplication coming from the group structure of $SU(2)$.

The R-matrix

Consider the matrix $R \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) = M_4(\mathbb{C})$ given by

$$R = \begin{pmatrix} R^{11}_{11} & R^{11}_{12} & R^{11}_{21} & R^{11}_{22} \\ R^{12}_{11} & R^{12}_{12} & R^{12}_{21} & R^{12}_{22} \\ R^{21}_{11} & R^{21}_{12} & R^{21}_{21} & R^{21}_{22} \\ R^{22}_{11} & R^{22}_{12} & R^{22}_{21} & R^{22}_{22} \end{pmatrix} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

This matrix can be used to describe $\mathcal{O}(SU_q(2))$ as follows.

Lemma

The algebra $\mathcal{O}(SU_q(2))$ is the universal algebra with generators u^i_j for $1 \leq i, j \leq 2$ and relations

$$\sum_{k,l=1}^{2} R^{ji}_{kl} u^k_m u^l_n = \sum_{k,l=1}^{2} u^i_k u^j_l R^{lk}_{mn}$$

for $1 \leq i, j, m, n \leq 2$, together with the quantum determinant relation

$$u^1_1 u^2_2 - q u^1_2 u^2_1 = 1.$$

The R-matrix

One can check that the matrix R satisfies the *quantum Yang-Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

in $M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$.

The quantum Yang-Baxter equation plays a role in statistical mechanics, and originally quantum groups were invented to study solutions to the quantum Yang-Baxter equation.

The theory of R-matrices also provides the link between quantum groups and knot invariants.

Some notation

- Fix $q = e^h \in (0,1)$.
- Let $\mathfrak{g}$ be a semisimple complex Lie algebra of rank N with Cartan matrix (a_{ij}).
- Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra.
- Let $\Delta = \Delta^+ \cup \Delta^-$ be the root system with simple roots $\alpha_1, \ldots, \alpha_N \subset \mathfrak{h}^*$.
- Let $(\ ,\ )$ be the bilinear form on $\mathfrak{h}^*$ obtained by rescaling the Killing form such that all short roots α satisfy $(\alpha, \alpha) = 2$.
- Set $d_i = (\alpha_i, \alpha_i)/2$ and $q_i = q^{d_i}$.
- Let $\varpi_1, \ldots, \varpi_N \in \mathfrak{h}^*$ be the fundamental weights.
- Let $\mathbf{P} = \bigoplus_{j=1}^N \mathbb{Z}\varpi_j$ and $\mathbf{R} = \bigoplus_{j=1}^N \mathbb{Z}\alpha_j$ be the weight and root lattices, respectively.
- Let $\mathbf{P}^+ = \bigoplus_{j=1}^N \mathbb{N}_0\varpi_j$ be the set of dominant integral weights.
- Let W be the Weyl group of $\mathfrak{g}$.

Quantized universal enveloping algebras

Definition

The quantized enveloping algebra $U_q(\mathfrak{g})$ is the algebra with generators K_λ for $\lambda \in \mathbf{P}$ and E_i, F_i for $i = 1, \ldots, N$ and relations

$$K_0 = 1,\ K_\lambda K_\mu = K_{\lambda+\mu}, \quad K_\lambda E_j K_\lambda^{-1} = q^{(\lambda,\alpha_j)} E_j,\ K_\lambda F_j K_\lambda^{-1} = q^{-(\lambda,\alpha_j)} F_j$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

for all $\lambda, \mu \in \mathbf{P}$ and all i, j, and the quantum Serre relations

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0.$$

Here the brackets denote q-binomial coefficients, which are certain deformations of ordinary binomial coefficients.

Quantized universal enveloping algebras

The algebra $U_q(\mathfrak{g})$ becomes a Hopf $*$-algebra in the following way.

The comultiplication $\hat{\Delta} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$, counit $\hat{\epsilon} : U_q(\mathfrak{g}) \to \mathbb{C}$, and antipode $\hat{S} : U_q(\mathfrak{g}) \to U_q(\mathfrak{g})$ are given by

$$\begin{aligned} \hat{\Delta}(K_\lambda) &= K_\lambda \otimes K_\lambda, \\ \hat{\Delta}(E_i) &= 1 \otimes E_i + E_i \otimes K_i \\ \hat{\Delta}(F_i) &= K_i^{-1} \otimes F_i + F_i \otimes 1, \end{aligned}$$

$$\hat{\epsilon}(K_\lambda) = 1, \qquad \hat{\epsilon}(E_j) = 0, \qquad \hat{\epsilon}(F_j) = 0,$$

$$\hat{S}(K_\lambda) = K_{-\lambda}, \qquad \hat{S}(E_j) = -E_j K_j^{-1}, \qquad \hat{S}(F_j) = -K_j F_j,$$

respectively. The $*$-structure is given by

$$E_i^* = K_i F_i, \qquad F_i^* = E_i K_i^{-1}, \qquad K_\lambda^* = K_\lambda.$$

Quantized universal enveloping algebras

Let us write $\hat{\Delta}^{\mathrm{cop}}$ for the copposite comultiplication on $U_q(\mathfrak{g})$, that is

$$\hat{\Delta}^{\mathrm{cop}}(X) = X_{(2)} \otimes X_{(1)} = \sigma(X_{(1)} \otimes X_{(2)}) = \sigma\hat{\Delta}(X)$$

where σ is the flip map.

One central features of the theory is that $U_q(\mathfrak{g})$ is *quasitriangular*.

This means that there exists a *universal* R*-matrix*, that is, an invertible element $\mathcal{R}$ in (a certain completion of) $U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ satisfying

$$\hat{\Delta}(X) = \mathcal{R}\hat{\Delta}^{\mathrm{cop}}(X)\mathcal{R}^{-1}$$

for all $X \in U_q(\mathfrak{g})$, and some further relations, in particular the quantum Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

Representation theory

Let $\pi_\lambda : U_q(\mathfrak{g}) \to \mathrm{End}(V)$ be a representation of $U_q(\mathfrak{g})$. For $\lambda \in \mathfrak{h}^*$ we define the weight space

$$V_\lambda = \{v \in V \mid \pi_\lambda(K_\alpha)(v) = K_\alpha \cdot v = q^{(\alpha,\lambda)} v \text{ for all } \alpha \in \mathbf{P}\}.$$

The representation V is called a weight representation if it is the direct sum of its weight spaces V_λ for $\lambda \in \mathfrak{h}^*$.

A representation of highest weight $\lambda \in \mathfrak{h}^*$ is a representation V with a cyclic vector $v_\lambda \in V$ such that

$$K_\alpha \cdot v_\lambda = q^{(\alpha,\lambda)} v_\lambda \qquad \text{and} \qquad E_i \cdot v_\lambda = 0 \quad \text{for all } i = 1, \ldots, N.$$

Theorem

For any dominant integral weight $\lambda \in \mathbf{P}^+$ there exists a finite dimensional irreducible $*$-representation $V(\lambda)$ of $U_q(\mathfrak{g})$ of highest weight λ, unique up to isomorphism. Moreover, every finite dimensional weight representation of $U_q(\mathfrak{g})$ is completely reducible.

Semisimple compact quantum groups

Let $e_1, \ldots, e_n$ be an orthonormal basis of $V(\lambda)$. Then we define linear functionals $u^i_j \in U_q(\mathfrak{g})^*$ by

$$u^i_j(X) = \langle e_i, \pi_\lambda(x) e_j \rangle.$$

The linear span of the functionals u^i_j for $1 \leq i, j \leq n$ is called the space of matrix coefficients of π_λ and denoted $C(\lambda)$.

Now let K be a simply connected semisimple compact Lie group and let $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ be the complexification of its Lie algebra.

We set

$$\mathcal{O}(K_q) = \bigoplus_{\lambda \in \mathbf{P}^+} C(\lambda).$$

Semisimple compact quantum groups

Given matrix coefficients $f, g \in \mathcal{O}(K_q)$ we define their product by

$$(fg)(X) = f(X_{(2)})g(X_{(1)}).$$

Then fg is again contained in $\mathcal{O}(K_q)$.

Moreover define a comultiplication $\Delta : \mathcal{O}(K_q) \to \mathcal{O}(K_q) \otimes \mathcal{O}(K_q)$, a counit $\epsilon : \mathcal{O}(K_q) \to \mathbb{C}$, and an antipode $S : \mathcal{O}(K_q) \to \mathcal{O}(K_q)$ by

$$\Delta(f)(X \otimes Y) = f(XY), \quad \epsilon(f) = f(1), \quad S(f)(X) = f(\hat{S}^{-1}(X))$$

and a $*$-structure by $f^*(X) = \overline{(f, \hat{S}^{-1}(X)^*)}$.

This turns $\mathcal{O}(K_q)$ into a Hopf $*$-algebra.

Semisimple compact quantum groups

Definition

Let $q \in (0,1)$ and let K be a simply connected semisimple compact Lie group. The C^*-algebra of functions $C(K_q)$ on the compact quantum group K_q is the enveloping C^*-algebra of $\mathcal{O}(K_q)$.

One verifies that $C(K_q)$ is a unital Hopf C^*-algebra with comultiplication induced from the comultiplication Δ of $\mathcal{O}(K_q)$. It automatically has a left and right invariant Haar state ϕ.

General theory produces a multiplicative unitary $W \in \mathcal{L}(\mathcal{H} \otimes \mathcal{H})$, where $\mathcal{H} = L^2(G)$ is the GNS-construction of ϕ, and the group C^*-algebra $C^*(K_q)$.

Proposition

The group C^*-algebra of K_q is

$$C^*(K_q) = \bigoplus_{\lambda \in \mathbf{P}^+} \mathcal{L}(V(\lambda)).$$