Tropical Geometry

- Many problems in Alg. Geo. are "invariant" under change of base field.
- Relevant for Mirror Symmetry: curve counting

Ex: (i) # lines in $\mathbb{P}^2_{\mathbb{C}}$ passing through 2 dist. pts P_1, P_2 equals 1..
(ii) # conics passing through 5 general points in $\mathbb{P}^2_{\mathbb{C}}$ equals 1.

- In gen'l: could ask for $N^{irr}_{g,d}$;
number of irreducible curves of genus g, degree d passing through $3d-1+g$ points in $\mathbb{P}^2_{\mathbb{C}}$ (gen'l pts)
(Turns out this is always finite #)

- For this could replace $\mathbb{C}$ with any alg. closed field with char. 0.
We'll from now on consider $K = \mathbb{C}\{\{t\}\}$, the field of Puiseux series.

- $\mathbb{C}\{\{t\}\} = \left\{ a(t) = \sum_{q \in \mathbb{Q}} a_q t^q \;\middle|\; a_q \in \mathbb{C}, \text{ subset of } q \in \mathbb{Q} \text{ with } a_q \neq 0 \text{ bounded from below, finite set of denominators.} \right\}$

In fact: $K = \bigcup_{n \geq 1} \mathbb{C}((t^{1/n})) = \mathbb{C}((t))^{alg}$

$val: \mathbb{C}\{\{t\}\}^* \longrightarrow \mathbb{R}$, $val(a(t)) = q$, where q is the lowest exponent appearing in expansion of $a(t)$

we extend to $K = \mathbb{C}\{\{t\}\}$ by setting $val(0) = \infty$

Then: 1) $val(a\cdot b) = val(a) + val(b)$

2) $val(a+b) \geq \min\{val(a), val(b)\} \quad \forall a, b$.

Image of v is $\mathbb{Q}$, the "value group".

- Now $T = (K^*)^n = \mathrm{Spec}\, K[x_1^{\pm 1}, -, x_n^{\pm 1}]$ is rk n torus $/K$.

- Coordinate wise valuation map

$v : (K^*)^n \longrightarrow \mathbb{R}^n$

$(a_1, -, a_n) \longmapsto (val(a_1), -, val(a_n))$

- Here one should think of $\mathbb{R}^n$ as $N_{\mathbb{R}}$, with N the lattice of 1-P.S. of T.

- "Philosophy" of tropical geo.:
transfer questions about subvarieties $Y \subset X_\Sigma$, X_Σ tor. var. $/K$ with $\Sigma \subset N_{\mathbb{R}}$ a fan
(for example curves $C \subset \mathbb{P}^2$)
to questions about $\mathrm{trop}(Y \cap T)$, by which we mean: closure in $\mathbb{R}^n$ of image of $Y \cap T$ under coord. val. map $v : T \to \mathbb{R}^n$

- <u>Turns out</u>: $\mathrm{trop}(Y \cap T)$ is a <u>polyhedral complex</u>, i.e. piecewise linear structure in $\mathbb{R}^n$, a "tropical variety"

- Mostly, we'll consider curves on toric surfaces. (e.g. $C \subset \mathbb{P}^2$) In this case:

$C \cap T$ is zero-locus $V(f)$ of Laurent poly.

$f = \sum_{m \in S} a_m z^m \in K[z_1^{\pm 1}, z_2^{\pm 1}]$, $m = (m_1, m_2) \in \mathbb{Z}^2 (= M)$
$z^m = z_1^{m_1} z_2^{m_2}$
$S \subset M$ finite, $a_m \in K$

<u>Question:</u> How to describe $\mathrm{trop}(V(f))$?

To answer this, it is very useful to introduce the "tropical semi-ring" $\mathbb{R}_{trop} = (\mathbb{R}, \oplus, \odot)$

- As a set, this is $\mathbb{R}$, with <u>addition</u>
  $a \oplus b := \min(a, b)$
  and <u>multiplication</u> $a \odot b = a + b$
- There is no additive inverse, but one can view "∞" as additive neutral elmt, since $\min\{a, \infty\} = a \quad \forall\ a \in \mathbb{R}$.
- $f = \sum_m a_m z^m$ yields "tropical function" $F = \mathrm{trop}(f)$

by
$$F(z_1, z_2) = \min_{m \in S} \left\{ \mathrm{val}(a_m) + \sum_{i=1}^{2} z_i \cdot m_i \right\}$$
$$= \min_m \left\{ \mathrm{val}(a_m) + \langle m, z \rangle \right\}$$

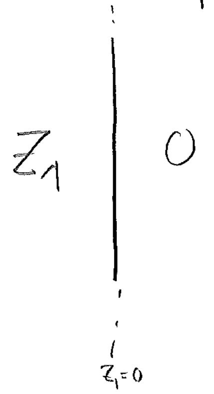
so $F : \mathbb{R}^2 \to \mathbb{R}$

Def: $V(\mathrm{trop}(f)) = \{(z_1, z_2) \in \mathbb{R}^2 \mid F \text{ not linear}\}$

this is called sometimes the "corner locus" of F.

Ex: $F = \min\{0, z_1\}$ $(= 0 \oplus (0 \odot z_1))$

this gives



Thm (Kapranov) $\mathrm{trop}(V(f)) = V(\mathrm{trop}(f))$

"Proof": Let $z = (z_1, z_2) \in (K^*)^2$ s.t. $f(z) = 0$, i.e.

$0 = \sum a_m z^m$. Then

$$(*) \quad \infty = \mathrm{val}(0) = \mathrm{val}\left(\sum a_m z^m\right)$$

But $\infty > \mathrm{val}(a_m z^m)$ $\forall$ m s.t. $a_m \neq 0$.

So $(*)$ can only be achieved if the minimum in $\{\mathrm{val}(a_m z^m)\}$ is obtained at least twice. Thus $(\mathrm{val}(z_1), \mathrm{val}(z_2)) \in V(\mathrm{trop}(f))$.

(Since $V(\mathrm{trop}(f))$ closed in $\mathbb{R}^2$, $\mathrm{trop}(V(f)) \subset V(\mathrm{trop}(f))$.)

The other inclusion is more complicated.

See [StML]

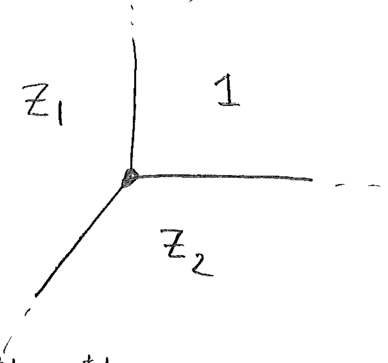
$\square$

Exercise: Let $f_1 = 1 + x_1$, $f_2 = 1 + t x_2$ in $K[x_1^{\pm 1}, x_2^{\pm 1}]$

Sketch $\mathrm{trop}(V(f_i))$, $i = 1, 2$.

Example: $F = 1 \oplus (0 \odot z_1) \oplus (0 \odot z_2)$

Then $V(F) \subset \mathbb{R}^2$ is



* Let $f = \sum a_m z^m \in K[z_1^{\pm 1}, z_2^{\pm 1}]$

We'll now explain how to "compute" $V(\mathrm{trop}(f))$.

- Let $P = \mathrm{Newt}(f) = \mathrm{conv}(m \mid a_m \neq 0)$ in $\mathbb{R}^2$, the Newtonpolytope of f.
- We'll abuse notation and write a_m instead of $\mathrm{val}(a_m)$.

  Put $\widetilde{P} = \mathrm{conv}(\{(m, a_m)\}) \subset \mathbb{R}^2 \times \mathbb{R}^1$,

  $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ projection onto first two coord's.

- Say that $F \leq \widetilde{P}$ is a lower face if

  $F = \mathrm{face}_v(\widetilde{P}) = \{ x \in \widetilde{P} \mid v \cdot x \leq v \cdot y \ \forall y \in \widetilde{P} \}$

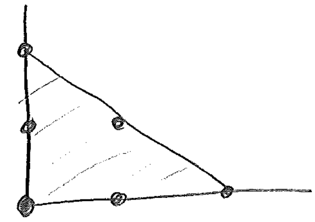
  with $v = (v_1, v_2, v_3)$ s.t. $v_3 > 0$.

- Geometrically, this means that $F = H \cap \widetilde{P}$, where H is hyperplane w/ slope v, s.t. "the rest" of $\widetilde{P}$ lies above H

<u>Def</u>: The regular subdivision of P, w.r.t $\{a_m\}_{m \in S}$, consists of the polytopes $\{\pi(F) \mid F \text{ is a lower face of } \tilde{P}\}$.

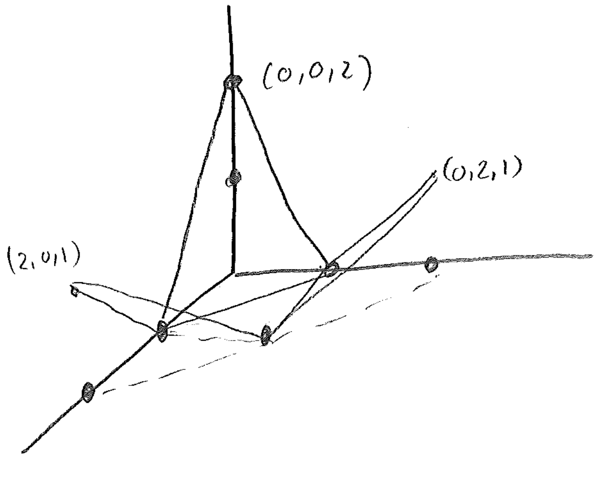
<u>Example</u>: $f = t z_1^2 + z_1 z_2 + t z_2^2 + z_1 + z_2 + t^2$

$\rightsquigarrow$ P is

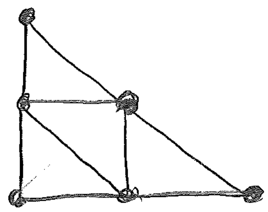


$\tilde{P}$ is $\operatorname{conv}\begin{pmatrix} [(2,0),1], [(1,1),0], [(0,2),1] \\ [(1,0),0], [(0,1),0], [(0,0),2] \end{pmatrix}$

The lower faces of $\tilde{P}$ are :



Giving regular subdiv.



- This can be used to compute trop(V(f)) as follows :

(*) trop(V(f)) is the 1-skeleton of the dual polyh. complex $\check{P}$ of the regular subdivision.

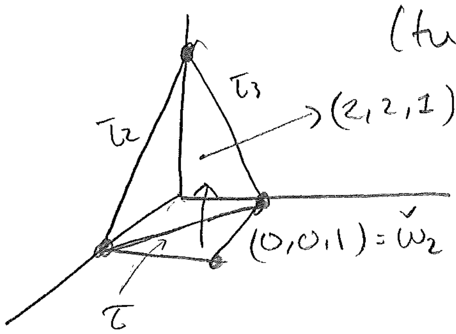
This is obtained by

$$F \rightsquigarrow \check{F} := \{ w \in \mathbb{R}^2 \mid \mathrm{face}_{(w,1)}(\tilde{P}) = F \}$$

Here, $\mathbb{R}^2$ should be interpreted as $N_{\mathbb{R}}$.

<u>Example:</u> With same f as in prev. ex., consider

(two of the lower facets)

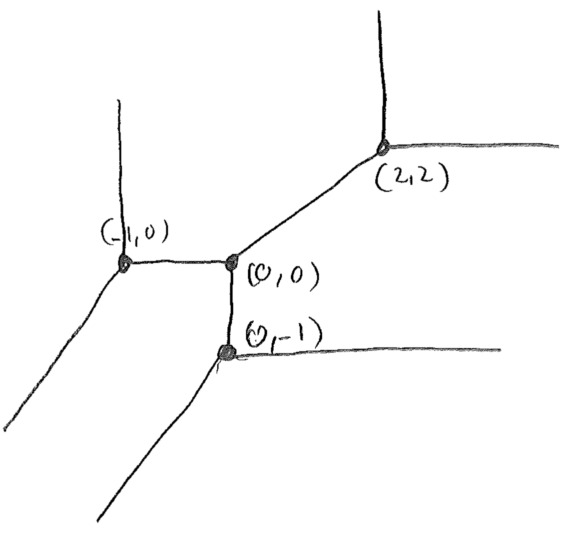


The vectors are "inward pt normal vectors", giving vertices $\check{\omega}_1, \check{\omega}_2$ of $\mathrm{trop}(V(f))$.

If τ_1 is the edge going from $(1,0,0)$ to $(0,1,0)$, any hyperplane containing τ_1, and supporting $\widehat{P}$ can only (when normalized $(\omega_1, \omega_2, 1)$ vary in a finite interval $\lambda \cdot (1,1)$, $0 < \lambda \leq 2$.

For τ_2, τ_3 we get <u>unbounded</u> lines.

<u>All in all:</u> $\mathrm{trop}(V(f))$ looks like



<u>Exercise:</u> Do the computation!

In fact:

The 2-cells of $\check{P}$ are the maximal domains of linearity of $\operatorname{trop}(V(f))$, and

$$\operatorname{trop}(f) = \bigoplus_{m \in P[0]} (a_m \odot z^m) = \min_m (a_m + \langle m, z \rangle)$$

where m runs over the vertices in the regular subdiv. of P.

To see (*), one argues as follows:

Let $\omega = (\omega_1, \dots, \omega_n, 1) \in N(F)$ for a lower face $F \leq \tilde{P}$

Consider $\displaystyle in_{\pi(\omega)}(f) := \sum_{W = a_m + \langle m, \pi(\omega) \rangle} t^{-a_m} \cdot a_m z^m$

$(W = \operatorname{trop}(f)(\pi(\omega)))$

- This is a sum of monomials whose exponents live in $\pi(F)$ (over other exponents $m' \notin \pi(F)$, can check that $t^{-a_{m'}} \cdot a_{m'}$ reduces to 0, because of $\omega \cdot x \leq \omega \cdot y$ in def. of F)
- All vertices in $\pi(F)$ appear with non-zero coefficient.
- So $\pi(\omega) = (\omega_1, \dots, \omega_n) \in \operatorname{trop}(V(f))$

  $\Leftrightarrow$ $(\omega_1, \dots, \omega_n) \in \pi(F)$ for a face F of $\tilde{P}$ such that $\pi(F)$ has more than one vertex.

Note also: if $W = \operatorname{trop}(f)(\pi(\omega))$ is the minimal value:

$$in_{\pi(\omega)}(f) = \sum_{\substack{m \\ W = a_m + \langle m, \pi(\omega) \rangle}} a_m t^{-\operatorname{val}(a_m)} z^m = t^{-W} \sum a_m t^{\langle m, \omega \rangle} z^m$$

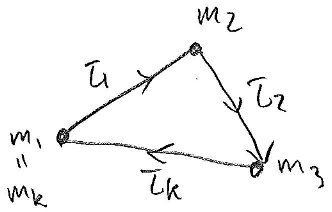
picks out exactly the monomials where min. is achieved, "the rest" reduce to zero.

Since the 2-cells of $\check{P}$ are the maximal domains of linearity of $\mathrm{trop}(V(f))$, and $\mathrm{trop}(f) = \bigoplus_{m \in P[0]} a_m \odot z^m = \min_{m \in P[0]} (a_m + \langle m, z \rangle)$,

follows that the edges of $\mathrm{trop}(V(f))$ separates domains of linearity for $\mathrm{trop}(f)$.

- This leads to the so-called "<u>balancing condition</u>"

* Let ω be a 2-cell i P, edges $\tau_1, \dots, \tau_k$
Choose orientation.

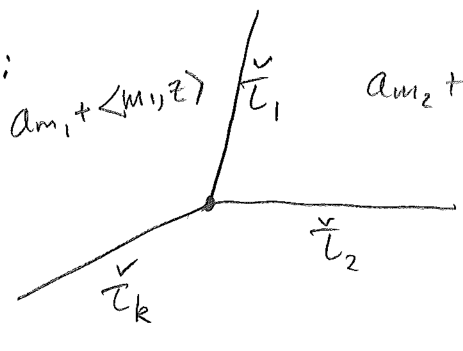


Then $m_{i+1} - m_i = W_i \cdot n_i$, where

W_i = weight of $\check{\tau}_i$ $(\in \mathbb{Z}_{>0})$

n_i = primitive tgt vector in direction of $\tau_i \in M$.

Dual picture:



<u>Then:</u> $\sum_i W_i \cdot n_i = 0$ ("Balancing condition")

<u>Note:</u> n_i is primitive <u>normalvector</u> to ray $\check{\tau}_i$.

$\forall z \in \check{\tau}_i$: $a_{m_i} + \langle m_i, z \rangle = a_{m_{i+1}} + \langle m_{i+1}, z \rangle$

$\rightsquigarrow \langle m_{i+1} - m_i, z \rangle = \text{const}$, $n_i = \frac{1}{W_i}(m_{i+1} - m_i)$

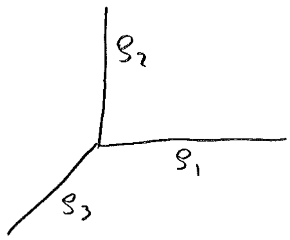
<u>Theorem:</u> Let $\Sigma \subset N_{\mathbb{R}} \approx \mathbb{R}^2$ be a weighted, balanced polyhedral (rational) cplx that is of pure dim'n 1.
Then $\exists$ tropical poly. F, with coeff. in $val(K^*)$ s.t. $\Sigma = V(F)$.

(rat'l : vertices, slopes are $\mathbb{Q}$-vectors)

<u>Pf:</u> (see Gross, or ML-St)

Applications of tropical geometry.

* Counting curves in $\mathbb{P}^2$ subject to point conditions.

<u>Recall:</u> $\mathbb{P}^2 \leftrightarrow$ fan $\Sigma_{\mathbb{P}^2}$  in $\mathbb{R}^2$

- If $\Gamma \subset \mathbb{R}^2$ is a rat'l, polyh. cplx, dim. 1, weighted, balanced, we say Γ is <u>in $\mathbb{P}^2$</u> if: each unbounded edge $E \in \Gamma_\infty^{[1]}$ is a translate of some ray $\rho \in \Sigma_{\mathbb{P}^2}$.

- Because of the balancing condition, there are (for some $d \in \mathbb{N}$) d edges, counted w/ mult., in each direction ρ_i, $i = 1, 2, 3$.

<u>Def:</u> d is the <u>degree</u> of Γ!

Def: The genus g of Γ is

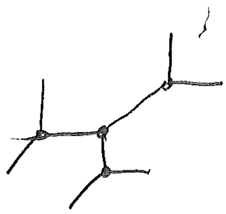
$b_1(\Gamma) = 1 + |\Gamma_1| - |\Gamma_0|$, where

Γ_1 = set of bounded edges

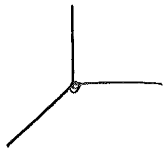
Γ_0 = set of vertices.

Hence: g is the # of cycles in Γ.

Ex:



$g=0$
$d=2$



$g=0$
$d=1$

* Let $N^{irr}_{g,d}$ be # of (irred.) curves in $\mathbb{P}^2$ of genus g, degree d, passing through $3d+g-1$ points $P_1, \ldots, P_{3d+g-1}$ in gen'l position.

(- One can show that this is a finite #.)

If $C \subset \mathbb{P}^2$ is one such, then

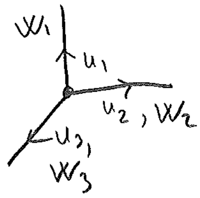
$trop(C \cap T) \subset \mathbb{R}^2$ is trop. curve of genus g, degree d passing through $trop(P_1), \ldots, trop(P_{3d+g-1})$ in $\mathbb{R}^2$

(- by genericity, points $P_i \in T$ $\forall i$)

* Mikhalkin proved that one can compute $N^{irr}_{g,d}$ by instead computing tropical curves "in $\mathbb{P}^2$".
  - This was the first major application of tropical geometry.

* One issue: One could have
  $\operatorname{trop}(C \cap T) = \operatorname{trop}(C' \cap T)$,
  so tropical curves should be counted with "<u>multiplicity</u>/weight"

* In fact
  $$Wt(\Gamma) = \prod_{V \in \Gamma^0} wt(V), \quad \text{where}$$



$$wt(V) = w_1 \cdot w_2 \, |\det(u_1, u_2)|$$
$$(= w_1 \cdot w_3 \, |\det(u_1, u_3)| \quad \text{etc})$$

* Similarly, can translate counting prob's for "stable maps" into count of tropical curves.

$\overline{M_{g,n}(\mathbb{P}^2, d)} \longleftrightarrow$ "parametrized trop. curves"
$h : (\Gamma, Ex_1, \ldots, Ex_n) \to \mathbb{R}^2$
Ex_i: marked unb. edge, $h|_{Ex_i} = \text{const}$
$h(\Gamma)$ trop. curve "in $\mathbb{P}^2$".

Some final words about tropical geom.

(A) Tropicalization beyond hypersurfaces:

$T = \operatorname{Spec} K[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, $X = V(I) \subset T$ (irred.) subvar.

$$\operatorname{trop}(X) := \bigcap_{f \in I} V(\operatorname{trop}(f)) \subset \mathbb{R}^n$$

- Can show: $\operatorname{trop}(X)$ = image of X under map $T \to \mathbb{R}^n = N_{\mathbb{R}}$
$(\operatorname{val}(), \dots, \operatorname{val}())$

<u>Moreover:</u> $\exists$ finite "tropical basis" $\{f_1, \dots, f_r\} = \mathcal{T}_I$

s.t. $\operatorname{trop}(X) = \bigcap_{i=1}^{r} V(f_i)$

$\mathcal{T}_I$ is trop. basis if it generates I, and, iff

$\forall \; w \in \Gamma^n$, Γ value grp. of $\operatorname{val}: K^* \to \mathbb{R}$,

$\operatorname{in}_w(I)$ contains a unit $\iff$ $\{\operatorname{in}_w(f) \mid f \in \mathcal{T}_I\}$ contains a unit.

- So $\operatorname{trop}(X)$ is finite intersection of "tropical hypers."

$\rightsquigarrow$ $\operatorname{trop}(X)$ is $\dim(X)$-dim'l polyh. cplx.
(connected, weighted, balanced)

(B) $X = X_\Sigma$ toric variety (say projective)

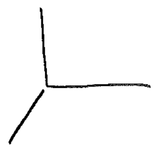
Then $X = \coprod_{\sigma \in \Sigma} T_\sigma$ $\quad$ T_σ torus orbit $\leftrightarrow \sigma$

Then - $\mathrm{Trop}(X) := \coprod_{\sigma \in \Sigma} \mathrm{Trop}(T_\sigma)$

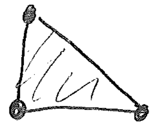
- If $Y \subset X$ subvar.,

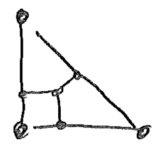
  $\mathrm{trop}(Y) := \coprod_{\sigma} \mathrm{trop}(Y \cap T_\sigma)$

- One can glue these together "in a nice way"

Ex: $X = \mathbb{P}^2$, $\Sigma =$ 

Then $\mathrm{trop}(\mathbb{P}^2) = \mathbb{R}^2 \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^1 \sqcup \mathbb{R}^0 \sqcup \mathbb{R}^0 \sqcup \mathbb{R}^0$

This glues to:  the polytope defining $\mathbb{P}^2$!

- If $L \subset \mathbb{P}^2$ a line, $\mathrm{trop}(L)$ is 

(C) This setup is functorial

If
$$\begin{tikzcd} X \arrow[r, "\phi"] & X' \\ Y \arrow[u, hook, "i"] \arrow[ur, hook, "j"'] & \end{tikzcd}$$
where X, X' toric var's, ϕ equiv. map

Then: $\mathrm{Trop}(\phi) : \mathrm{Trop}(Y, i) \to \mathrm{Trop}(Y, j)$

- For each i, $\exists$ proper, cont. map

  $\pi_i : Y^{an} \longrightarrow \mathrm{Trop}(Y, i)$

  Y^{an} Berkovich analytif. of Y.

Exercise: Let X_Σ be toric surface $/\mathbb{K}$, defined by fan $\Sigma \subset N_{\mathbb{R}} \approx \mathbb{R}^2$

Let $u_\rho \in N$ be the (primitive) ray gen's.

We say that a tropical curve (rational, weighted, balanced) is "in X_Σ" if $\forall\ E \in \Gamma'_\infty$, E is a translate of ρ, for some $\rho \in \Sigma[1]$ ray.

Let $\mathbb{Z}^N$ be free ab. grp. with gen's t_ρ, $\forall\ \rho \in \Sigma(1)$.

The degree $\deg(\Gamma) := \sum d_\rho t_\rho$, where

d_ρ = # unbounded edges in direction of ρ (counted w/ mult)

1) Let $r: \mathbb{Z}^N \to N$ be defined by $t_\rho \mapsto u_\rho$.

Show that $r(\deg(\Gamma)) = 0$.

2) For $\mathbb{P}^2$, show that if Γ in $\mathbb{P}^2$, then

$\deg(\Gamma) = d(t_{\rho_1} + t_{\rho_2} + t_{\rho_3})$

(Hint: use balancing cond.)

References:

M. Gross : "Tropical Geometry and Mirror Symmetry" (Book)

MacLagan, Sturmfels : "Introduction to Tropical Geometry" (Book)

G. Mikhalkin : "Enumerative Tropical Algebraic Geometry in $\mathbb{R}^2$"

A. Gathmann : "Tropical Algebraic Geometry"

Cox, Little, Schenk : Toric Varieties (Book)

V. Batyrev : "Dual Polyhedra and Mirror Symmetry ...."