Lecture notes: Toric and Tropical Geometry

(I) Toric Varieties

1) We (usually) work over $\mathbb{C}$. $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$

- Algebraic torus of rank $n \geq 0$:

$$T = (\mathbb{C}^*)^n = \operatorname{Spec} \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

- Groupstructure: $T \times T \to T$

$$(t_1, \dots, t_n), (t_1', \dots, t_n') \mapsto (t_1 \cdot t_1', \dots, t_n \cdot t_n')$$

- Monomials in coord. ring $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ of T form lattice: $m \in \mathbb{Z}^n \leftrightarrow x^m = x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}$

  This is denoted M ($\cong \mathbb{Z}^n$), can be seen as <u>character lattice</u> of T, $x^m : T \to \mathbb{C}^*$

$$(t_1, \dots, t_n) \mapsto t_1^{m_1} \cdot \ldots \cdot t_n^{m_n}.$$

- Dual lattice $N := \operatorname{Hom}(M, \mathbb{Z})$

  1-parameter subgrps of T: $u = (u_1, \dots, u_n) \in N$

$$\leftrightarrow \quad \lambda^u : \mathbb{C}^* \to T$$
$$t \mapsto (t^{u_1}, \dots, t^{u_n})$$

  N determines T: $N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong T$

$$u \otimes t \mapsto \lambda^u(t)$$

- Bilinear pairing $\langle \, , \, \rangle : M \times N \to \mathbb{Z}$

$$\langle m, n \rangle = \sum m_i \cdot n_i$$

  Extends to $\langle \, , \, \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \to \mathbb{R}$ (scalar product) where $M_{\mathbb{R}} = M \otimes \mathbb{R}$ ($= \mathbb{R}^n$) etc.

Let X be (irreducible) variety $/\mathbb{C}$

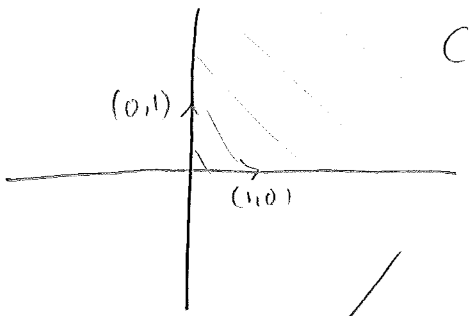
Def: X is toric variety if
(1) contains torus T as Zar. dense open subset
(2) Action $T\times T \to T$ extends to action $T \times X \to X$
$(t,x) \mapsto t\cdot x$

2) Toric var's from fans $\Sigma \subset N_{\mathbb{R}}$

Def: Convex polyhedral cone (in $N_{\mathbb{R}}$) is
$$\sigma = \mathrm{Cone}(S) = \Big\{ \sum_{u\in S} \lambda_u \cdot u \;\Big|\; \lambda_u \in \mathbb{R}_{\geq 0} \Big\}, \quad S \subseteq N_{\mathbb{R}} \text{ finite set.}$$
- σ is rational if $S \subset N$.
- σ is strongly convex if $\sigma \cap \{-\sigma\} = \{0\}$ (0 is only vertex)

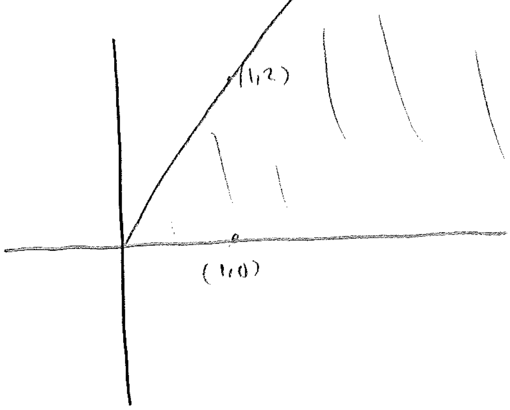
! We'll only consider rat'l, str. conv. cones, so we simply talk about cones.
- $\dim \sigma$ = dimension of the linear span of σ.

Ex1: (a)



$\mathrm{Cone}((1,0),(0,1)) \subset \mathbb{R}^2$

(b)



$\mathrm{Cone}((1,0),(1,2))$

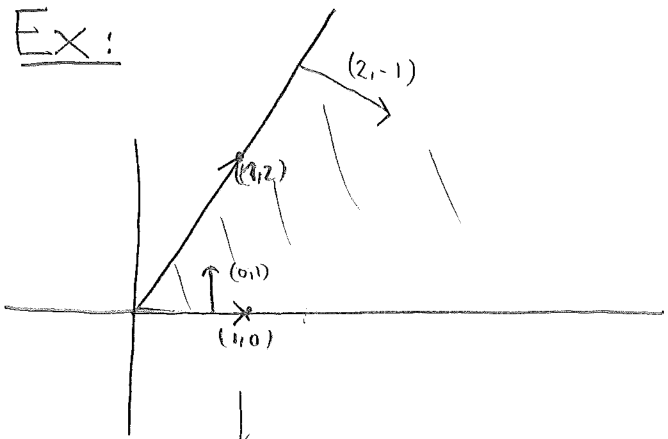
<u>Def</u>: If $\sigma \subset N_{\mathbb{R}}$ is a cone, the <u>dual cone</u> is $\check{\sigma} = \{ m \in M_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \ \forall\, u \in \sigma \}$

(It is again a cone)

- Every $m \in M_{\mathbb{R}}$ gives
- Hyperplane $H_m = \{ u \in N_{\mathbb{R}} \mid \langle m, u \rangle = 0 \}$
- (Closed) halfspace $H_m^+ = \{ u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq 0 \}$
- If $\sigma \subseteq H_m^+$, H_m^+ is called supporting halfspace (so $m \in \check{\sigma}$). If so:
  Then $\tau := \sigma \cap H_m$ is a <u>face</u> (is again cone)
- If $\dim \tau = \dim \sigma - 1$, τ is a <u>facet</u>
  $\dim \rho = 1$, ρ is an <u>edge</u> (or <u>ray</u>)

<u>Fact</u>: If $m_1, \dots, m_s$ generate the cone $\check{\sigma}$, then $\sigma = H_{m_1}^+ \cap \dots \cap H_{m_s}^+$ (and vice versa)

<u>Ex</u>:

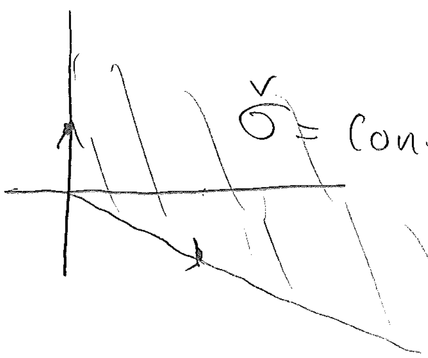


$\sigma = \text{Cone}((1,0),(1,2))$

Inward pointing facet normals are $(0,1)$, $(2,-1)$, so

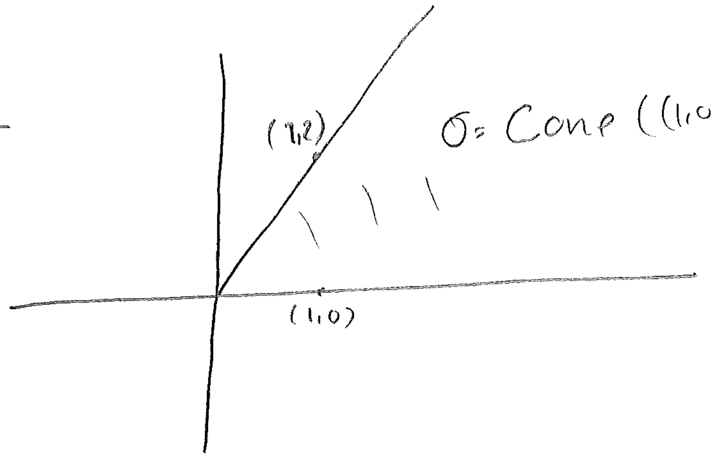
$\sigma = H^+_{(2,-1)} \cap H^+_{(0,1)}$

so



$\check{\sigma} = \text{Cone}((2,-1),(0,1))$

Ex:



$\sigma = \mathrm{Cone}((1,0),(1,2))$

The faces are: the vertex $(0,0)$
the edges $\mathrm{Span}((1,0))$, $\mathrm{Span}(1,2)$
and σ itself.

- If ρ is a ray, then the first lattice point on $\rho \cap N$ is denoted u_ρ (the unique generator of the semigroup $\rho \cap N$)

$*$ Put $\mathbb{C}[\sigma^\vee \cap M] \subset \mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

Fact: This is a finitely gen'd $\mathbb{C}$-alg., and is moreover an integral domain.

- Put $X_\sigma = \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$

Fact: This is an affine toric variety.
- This can be derived from the inclusion
$\mathbb{C}[\sigma^\vee \cap M] \subset \mathbb{C}[M]$

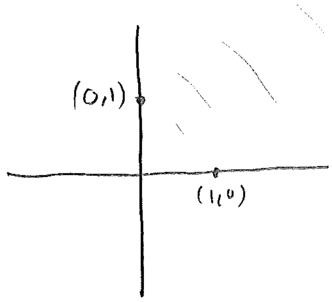
Roughly: This translates into the fact that
$T \subseteq X_\sigma$ (Z-dense), and

$$T \times T \to T \quad \longleftrightarrow \quad \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[M] \longleftarrow \mathbb{C}[M]$$
$$X^m \otimes X^m \longleftarrow\!\shortmid\; X^m$$

and one checks that $\mathbb{C}[\check{\sigma} \cap M]$ is stable under this action, which means that

$$\begin{array}{ccc} T \times T & \to & T \\ \cup & & \cup \\ T \times X_\sigma & \to & X_\sigma \end{array}$$

Ex:



$\sigma = \text{Cone}((1,0),(0,1))$

$\check{\sigma} = \text{Cone}((1,0),(0,1))$

$\mathbb{C}[\check{\sigma} \cap M] = \mathbb{C}[x_1, x_2]$,

so $X_\sigma \cong \mathbb{A}^2$.

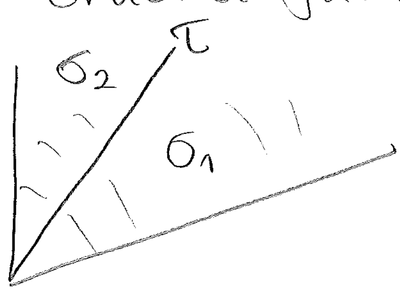
Thm: X_σ is smooth $\Leftrightarrow$ $\{u_\rho \mid \rho \text{ ray of } \sigma\}$ forms (part of) $\mathbb{Z}$-basis of N

Def: A fan Σ in $N_{\mathbb{R}}$ is collection of cones $\{\sigma \subset N_{\mathbb{R}}\}$, $\Sigma = \bigcup \sigma$ s.t.

- If $\tau \preccurlyeq \sigma$ face, then $\tau \in \Sigma$
- $\sigma \cap \sigma'$ is a face of both σ, σ' $\forall$ cones in Σ.

* This gives a toric variety $X_\Sigma := \bigcup_{\sigma \in \Sigma} X_\sigma$

- The crucial fact is the following: Given



$\tau = H_m \cap \sigma_1 = H_m \cap \sigma_2$

(for any $m \in \text{Relint}(\check{\sigma}_1 \cap (-\sigma_2)^\vee)$)

by "Separation Lemma"

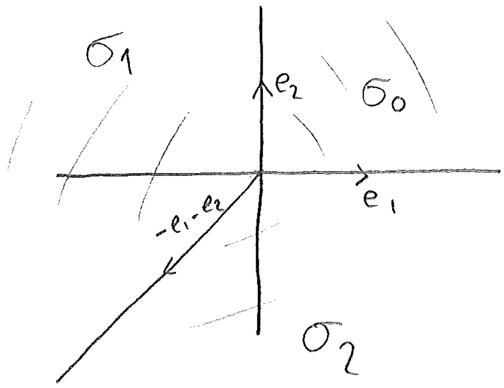
Then: $X_{\sigma_2} \supseteq (X_{\sigma_2})_{x^{-m}} = X_\tau = (X_{\sigma_1})_{x^m} \subseteq X_{\sigma_1}$

This means that we can glue $X_{\sigma_1} \cup_{X_\tau} X_{\sigma_2}$

$\rightsquigarrow$ X_Σ is a <u>toric variety</u> (the glueing is compatible with torus embeddings $T \subset X_\sigma$, and the T-action)

<u>Ex:</u>

(a)



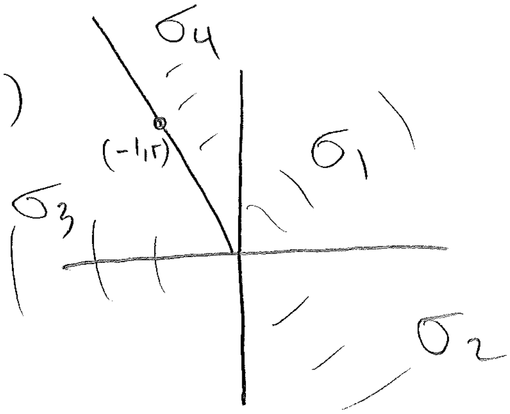
$\check{\sigma}_0 \cap M = \mathbb{N}^2 \subset \mathbb{Z}^2$

so $X_{\sigma_0} = \operatorname{Spec} \mathbb{C}[\mathbb{N}^2] = \mathbb{A}^2$

Likewise, can check $X_{\sigma_1} = \mathbb{A}^2$, $X_{\sigma_2} = \mathbb{A}^2$, and $X_\Sigma = \mathbb{P}^2$

(b) If $\Sigma \subset N_\mathbb{R} = \mathbb{R}^n$, and Σ is the fan induced by $\{e_1, \ldots, e_n, -e_1 - \ldots - e_n\}$, then $X_\Sigma = \mathbb{P}^n$

(c)



$\rightsquigarrow$ $X_\Sigma = \mathcal{H}_r$ is the r-th Hirzebruch surface.

<u>Exercise:</u> Check that $X_\Sigma = \mathbb{P}^2$ in (a)

* Further properties:

- X_Σ is proper $\Leftrightarrow$ $|\Sigma| = N_{\mathbb{R}}$
(support of Σ)

To see this, one proves that properness is $\Leftrightarrow$

$$\lim_{t\to 0} \lambda^u(t) \text{ exists in } X_\Sigma \quad \forall\, u \in N$$

(i.e., suffices to check valuative criterion for 1-PS in this case)

On the other hand, one shows: $\forall\, \sigma \in \Sigma$

$$u \in \sigma \Leftrightarrow \lim_{t\to 0} \lambda^u(t) \text{ exists in } X_\sigma$$

- If one takes $u \in \operatorname{Relint}(\sigma)$, then $\lim_{t\to 0} \lambda^u(t) = \gamma_\sigma$,
(interior of σ in its linear span)
independently of choice of such u.

Then $O(\sigma) = T_N \cdot \gamma_\sigma \subseteq X_\Sigma$ is the torus orbit corresponding to σ.

<u>Fact</u>: Let N_σ be the sublattice of N spanned by $\sigma \cap N$, let $N(\sigma) = N/N_\sigma$ (is again lattice).

Then: $O(\sigma) \cong T_{N(\sigma)}$, where $T_{N(\sigma)} = N(\sigma) \otimes_{\mathbb{Z}} \mathbb{C}^*$.

<u>Thm</u>:
$$\{\sigma \in \Sigma\} \xleftrightarrow{1:1} \{T_N\text{-orbits in } X_\Sigma\}$$
(dimension reversing)

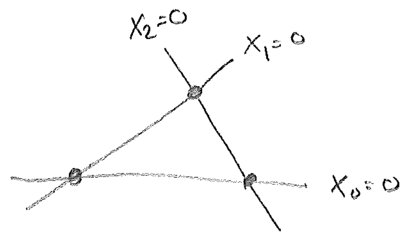
So X_Σ has a stratification by <u>tori</u>!

Ex: $\mathbb{P}^2 = \operatorname{Proj} \mathbb{C}[x_0, x_1, x_2]$

$\longleftrightarrow$ 

in $\mathbb{R}^2$

Torus orbits are:



- dense torus is complement of the lines.
- 1-dim'l orbits: each line, minus the two points • • on it.
- 0-dim'l orbits: the three pts. •, •, •

* If $\begin{matrix} N_1 & \xrightarrow{\phi} & N_2 \\ \cup & & \cup \\ \Sigma_1 & & \Sigma_2 \end{matrix}$ is $\mathbb{Z}$-linear map s.t. each $\sigma_1 \in \Sigma_1$ is mapped into a cone $\sigma_2 \in \Sigma_2$

then: $\exists!$ $X_{\Sigma_1} \to X_{\Sigma_2}$ morphism of toric var's compatible w/ torus actions.

* <u>Canonical divisor</u> of toric variety X_Σ:

$$K_{X_\Sigma} = -\sum_{\rho \in \Sigma(1)} D_\rho$$

Here D_ρ is the codim 1 torusorbit corresponding to the ray ρ under the orbit-cone correspondence!

3) Projective toric varieties, Polytopes

- By a <u>polytope</u> $P \subset M_{\mathbb{R}}$, we shall mean

$$P = \mathrm{Conv}(S), \text{ where } S \subset M \text{ finite set.}$$

- We shall (usually) assume $0 \in \mathrm{int}(P)$ and P full dim'l.

  ↳ strictly speaking: this is a lattice polytope

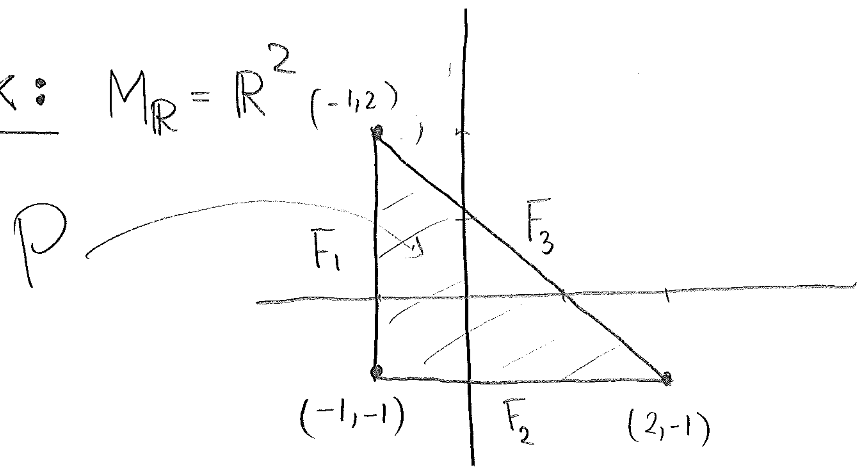
- For $u \in N_{\mathbb{R}}$, $b \in \mathbb{R}$,

$$H_{u,b} = \{ m \mid \langle m, u \rangle = b \}$$

$$H^+_{u,b} = \{ m \mid \langle m, u \rangle \geq b \}$$

- So like for cones, we can speak of supporting hyperplanes, supporting halfspaces, faces, facets, edges, vertices of P.

Ex: $M_{\mathbb{R}} = \mathbb{R}^2$



Inward facing facet normals are:

$u_{F_1} = (1,0)$, $a_{F_1} = 1$

$u_{F_2} = (0,1)$, $a_{F_2} = 1$

$u_{F_3} = (-1,-1)$, $a_{F_3} = 1$

Then $P = \bigcap_{i=1}^{3} H^+_{u_{F_i}, -a_{F_i}}$

In general: can write

$P = \bigcap_{F \text{ facet}} H_F^+$, where $H_F^+ := H^+_{u_F, -a_F}$,

and we have "normalized" so that u_F is inwards pointing facet normal, $u_F \in N$ primitive, and $a_F \in \mathbb{Z}_{>0}$.

Note: since we assume $0 \in \mathrm{int}(P)$, then by

$P = \{ m \mid \langle m, u_F \rangle \geq -a_F \mid F \text{ facet of } P \}$,

the a_F-s are > 0,

and for a vertex $v \in F$, $\langle v, u_F \rangle = -a_F$, so a_F is an integer.

Def: • The normal fan Σ_P to P is $\bigcup_{\substack{Q \preceq P \\ \text{face}}} \sigma_Q$, where $\sigma_Q = \mathrm{Cone}(u_F \mid Q \preceq F, F \text{ facet})$

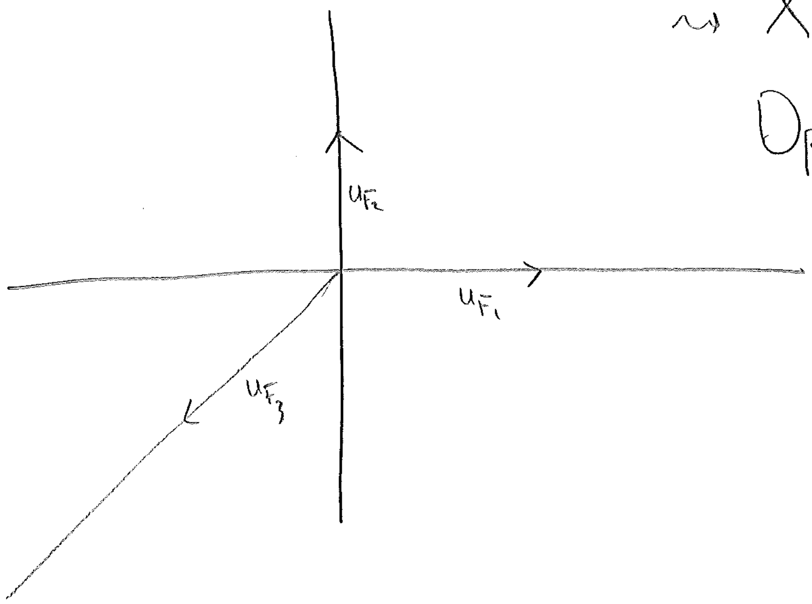
• $X_P :=$ the toric variety corresponding to $\Sigma_P \subset N_{\mathbb{R}}$.

General facts: - $|\Sigma_P| = N_{\mathbb{R}}$, so X_{Σ_P} proper.

- $D_P := \sum_{F \text{ facet}} a_F D_F$ is ample Cartier-div.

Hence: X_P is projective variety.

Ex: From P in previous example,
we get normal fan



$\rightsquigarrow X_P = \mathbb{P}^2$

$D_P = L_0 + L_1 + L_2$,

where $L_i = \{x_i = 0\}$
for $i = 0, 1, 2$.

- $\forall\ k \geq 0$,

$$L(kP) := \Gamma(X_P, \mathcal{O}_{X_P}(k D_P)) \cong \bigoplus_{m \in (kP) \cap M} \mathbb{C} \cdot \chi^m$$

  - These are the <u>effective</u> divisors lin. eq. to $k \cdot D_P$

(- also: k-th graded piece of hom. coord. ring of X_P)

Batyrev's construction

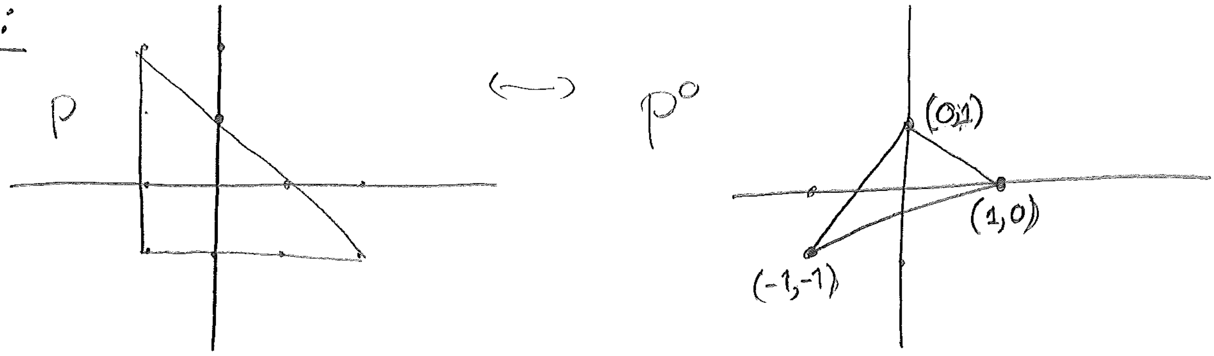
- Let P be polytope s.t. all a_F-s are 1, i.e.

$$P = \{ m \in M_{\mathbb{R}} \mid \langle m, u_F \rangle \geq -1 \ , \ F \text{ facet} \}.$$

Then $P^\circ = \{ u \in N_{\mathbb{R}} \mid \langle m, u \rangle \geq -1 \ \forall\, m \in P \} = \operatorname{Conv}(u_F \mid F \text{ facet})$ is again a (lattice) polytope, and $(P^\circ)^\circ = P$.

Such polytopes are called "<u>reflexive polytopes</u>"

<u>Ex:</u>



* <u>Note:</u> If P is such a reflexive polytope, then

$$D_P = \sum_{F \text{ facet}} D_F = -K_{X_P}$$

- Take "general" $f \in L(P) = H^0(X_P, \mathcal{O}_{X_P}(D_P)) = \bigoplus_{m \in P \cap M} \mathbb{C} \cdot \chi^m$

$Z_f = \{ f = 0 \} \subset T$ is a hypersurface in T.

Put $\overline{V} := \{ \text{closure of } Z_f \}$ in X_P

<u>Prop.</u>: $\overline{V}$ is a Calabi-Yau variety.

Proof:

Since the divisor $\overline{V} \in |-K_{X_p}|$, i.e. is eff. div. linearly eq. to $-K_{X_p}$, the adjunction formula states:

Let $\hat{\Omega}^{n-1}_{\overline{V}}$ be can. sheaf of $\overline{V}$

$\hat{\Omega}^{n}_{X_p}$ be can. sheaf of X_p $\quad$ ($\dim X_p = n$)

Then $$\hat{\Omega}^{n-1}_{\overline{V}} \cong \left(\hat{\Omega}^{n}_{X_p} \otimes \mathcal{O}_{X_p}(\overline{V})\right)\Big|_{\overline{V}} = \mathcal{O}_{\overline{V}},$$

since $\overline{V} = -K_{X_p}$, i.e., $\mathcal{O}_{X_p}(\overline{V}) = \mathcal{O}(-K)$, and

$\hat{\Omega}^{n}_{X_p} = \mathcal{O}(K)$.

Secondly: Since the ideal sheaf defining $\overline{V} \subset X_p$ is $\mathcal{O}_{X_p}(-\overline{V}) \cong \hat{\Omega}^{n}_{X_p}$, get from s.e.s

$$0 \to \mathcal{O}_{X_p}(-\overline{V}) = \hat{\Omega}^{n}_{X_p} \to \mathcal{O}_{X_p} \to \mathcal{O}_{\overline{V}} \to 0$$

long exact sequence in cohomology:

$$\cdots \to H^k(X_p, \mathcal{O}_{X_p}) \to H^k(\overline{V}, \mathcal{O}_{\overline{V}}) \to H^{k+1}(X_p, \hat{\Omega}^{n}_{X_p}) \to \cdots$$

But $H^k(X_p, \mathcal{O}_{X_p}) = 0$ for $k > 0$ (gen'l result for toric var's)

$H^{k+1}(X_p, \hat{\Omega}^{n}_{X_p}) = 0$ for $k < n-1$ (By Serre duality)

It follows that $H^k(\overline{V}, \mathcal{O}_{\overline{V}}) = 0$ for all $0 < k < n-1$. Hence $\overline{V}$ is Calabi-Yau.

□

- Of course, X_P and $\bar{V}$ may not be smooth varieties. But they have rather mild singularities. (By results of M. Reid)

- We'll assume now $\dim P = 4$.

- One can find a suitable subdivision $\Sigma \to \Sigma_P$ of the fan Σ_P (so called maximal projective subdiv.) so that the induced proper birat'l map

$\varphi : X_\Sigma \longrightarrow X_P$ satisfies

- $\varphi^* K_{X_P} = K_{X_\Sigma}$ (a "crepant" resolution)

- X_Σ has only fin. many isolated sing. pts.

- $V = \varphi^{-1}(\bar{V})$ is smooth 3-fold ($V \to \bar{V}$ crepant res. also)

Moreover: By repeating the argument in Prop., since $V \in |-K_{X_\Sigma}|$; V is a CY-variety (smooth).

Conclusion: V CY 3-fold $\left(H^k(V, \mathcal{O}_V) = H^{0,q} = 0,\ q = 1,2 \right)$

But by reflexivity, could do precisely same construction for P° instead, this would give CY 3-fold V°.

Thm (Batyrev) For V, V° the following holds:

$h^{1,1}(V) = h^{2,1}(V^\circ)$, $h^{2,1}(V) = h^{1,1}(V^\circ)$

i.e. since Hodge diamond for CY-3fold,

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{2,1} & & 1 \\ & 0 & & h^{2,2} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

(and $h^{1,1} = h^{2,2}$, $h^{1,2} = h^{2,1}$)

this means that V and V° are mirror duals (in at least a weak sense)

Here's a sketch of the proof for why this holds:

In fact, Batyrev proved that

$$h^{1,1}(V) = \ell(P^{\circ}) - 5 - \sum_{\substack{\Gamma^{\circ} \\ \text{codim 1} \\ \text{face of } P^{\circ}}} \ell^*(\Gamma^{\circ}) + \sum_{\substack{\Theta^{\circ} \\ \text{codim 2} \\ \text{faces of } P^{\circ}}} \ell^*(\Theta^{\circ})\,\ell^*(\hat{\Theta}^{\circ})$$

Here $\hat{\Theta}^{\circ}$ dual face in P of Θ°

$\ell^*(\Delta) =$ interior lattice point of a polytope Δ.

Similar formula for $h^{2,1}(V)$,

$$h^{2,1}(V) = \ell(P) - 5 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta) \cdot \ell^*(\hat{\Theta})$$

We'll explain the formula $h^{1,1}(V)$

Computation of $h^{2,1}(V)$

* A gen'l formula of Danilov-Khovanskii states

$$h^{n-2,1}(Z_f) = \ell(P) - n - 1 - \sum_{\substack{\Theta \\ \operatorname{codim}\Theta = 1}} \ell^*(\Theta)$$

For us $n=4$. For non-compact Y, $h^{p,q}(Y)$ Hodge-Deligne numbers.

<u>Def</u>: Y cplx variety,

$$e_c^{p,q}(Y) = \sum_{i \geq 0} (-1)^i h^{p,q}(H_c^i(Y)),$$ where $H_c^i(Y)$ cplx coh., w/ compact support.

- Behaves well under <u>products</u> $Y = Y' \times Y''$

$$e_c^{p,q}(Y) = \sum_{(p'+p'',\, q'+q'')=(p,q)} e_c^{(p',q')}(Y') \cdot e_c^{(p'',q'')}(Y'')$$

<u>Stratifications</u>: If $Y = \coprod_j Y_j$, where Y_j locally closed smooth subvar. $\overline{Y_j}$ = union of strata

$$e_c^{p,q}(Y) = \sum_j e_c^{p,q}(Y_j)$$

Now: V is smooth, proper, hence

$H_c^i(V) = H^i(V)$, and has pure H.S. of weight i.

Thus $e_c^{2,1}(V) = (-1)^3 h^{2,1}(V)$

so suffices to compute this one!

$\varphi : V \to \overline{V}$ is proper birat'l morphism

Moreover: $V = \bigcup_{\substack{\Theta \subset P \\ \text{faces}}} \varphi^{-1}(Z_{f,\Theta})$,

where $Z_{f,\Theta} = \overline{V} \cap (\text{toric stratum corresp. to } \Theta)$

(one has to choose f so that these are smooth, this is part of the genericity condition)

Each $\varphi^{-1}(Z_{f,\Theta})$ has stratification by smooth affine var's $\cong Z_{f,\Theta} \times (\mathbb{C}^*)^k$,

for varying k.

- If $\Theta \neq P$, then $e^{2,1}(Z_{f,\Theta} \times (\mathbb{C}^*)^k) \neq 0$ only if $\dim \Theta = 2$, $k = 1$.
- Can show: for each Θ with $\operatorname{codim} \Theta = 2$, gets $\ell^*(\Theta^\circ)$ strata isom. to $Z_{f,\Theta} \times \mathbb{C}^*$,

and
$$e_c^{2,1}(Z_{f,\Theta} \times \mathbb{C}^*) = \underbrace{e_c^{1,0}(Z_{f,\Theta})}_{(-1)^{n-3}\ell^*(\Theta)} \cdot \underbrace{e_c^{1,1}(\mathbb{C}^*)}_{1}$$

(By D-K again)

This finally gives the formula of Batyrev for $h^{2,1}(V)$. Similar arguments for $h^{1,1}(V)$.