The Lie Lie algebra

Jim Conant

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  - and antipode $S\colon H \to H$.

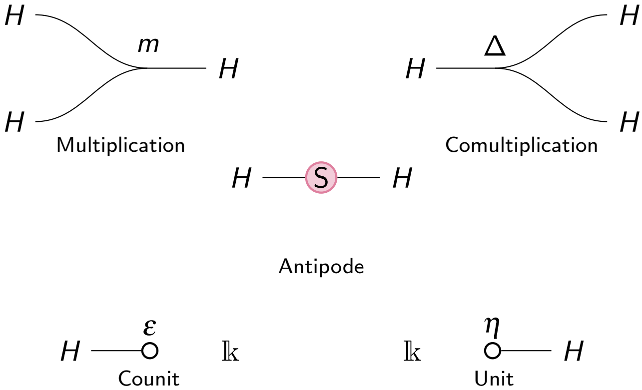

Figure: Hopf algebra operations depicted graphically, read from left to right.

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- Let $F_n = \langle x_1, \ldots x_n \rangle$. Given $\phi \in \text{Aut}(F_n)$ represent the transformation $F_n^n \to F_n^n$ given by $(x_1, \ldots, x_n) \mapsto (\phi(x_1), \ldots, \phi(x_n))$ as a composition of permutations of coordinates, multiplication of two coordinates, doubling of a coordinate and inversion of a coordinate. Now think of these operations instead as operations of the Hopf algebra.

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- For example suppose $\phi(x_1) = x_2^2 x_1^{-1}$ and $\phi(x_2) = x_1 x_2^{-1}$. Then

$$(a \otimes b) \cdot \phi = b_{(1)} b_{(2)} S(a_{(1)}) \otimes a_{(2)} S(b_{(3)}).$$

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Our main object of study

Let

$$\mathscr{H}_r(H) := \begin{cases} (\mathrm{Id} - S)H/[H,H] & r = 1 \\ H^{2r-3}(\mathrm{Out}(F_r); \overline{H^{\otimes r}}) & r \geq 2. \end{cases}$$

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1. $\mathscr{H}_r(\mathrm{Sym}(V))$ computes the abelianization of a Lie algebra $\mathfrak{h}^+(V)$, and can be used to construct homology classes in $H_*(\mathrm{Out}(F_n); \mathbb{k})$.
2. $\mathscr{H}_r(T(V))$ is the target of an invariant of the cokernel of the Johnson homomorphism.

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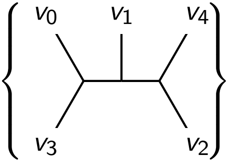
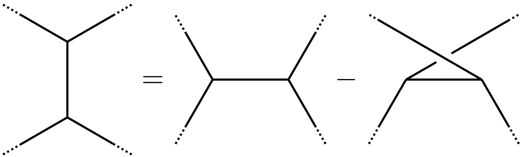
- Let $\mathfrak{h}^+(V)$ be generated by derivations of positive degree.
- $\mathfrak{h}(V)$ is well-known to be isomorphic to the space of Lie spiders. These are trivalent trees with univalent vertices labeled by elements of V, modulo orientation, IHX and multilinearity relations.

Spider interpetation of $\mathfrak{h}(V)$

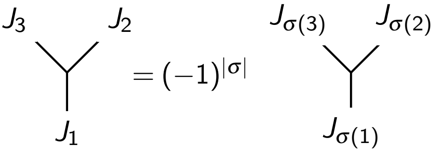
$$\mathfrak{h}_3(V) = \Bbbk \left\{ \quad \right\} / \mathrm{IHX} + \mathrm{AS} + \mathrm{MultiLin}$$



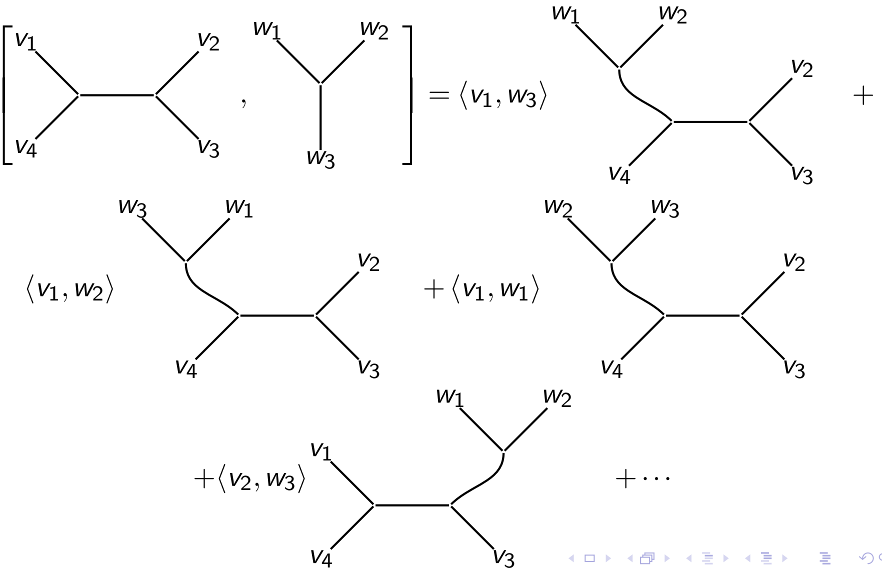
1. IHX:



2. AS:



The bracket map on spiders



Relation to homology of $\mathrm{Out}(F_n)$

Let $\cdots \leftarrow V_n \leftarrow V_{n+1} \leftarrow \cdots$ be a standard sequence of symplectic vector spaces V_n of dimension $2n$.

Theorem (Kontsevich)

$$\lim_{n\to\infty} PH^*(\mathfrak{h}^+(V_n))^{\mathrm{Sp}} \cong \bigoplus_{r\geq 2} H_*(\mathrm{Out}(F_r);\mathbb{k})$$

In particular the abelianization gives rise to potential homology classes in $\mathrm{Out}(F_r)$:

$$\lim_{n\to\infty} PH^*(\mathfrak{h}^+_{\mathrm{ab}}(V_n))^{\mathrm{Sp}} \to \bigoplus_{r\geq 2} H_*(\mathrm{Out}(F_r);\mathbb{k})$$

Theorem (Conant-Kassabov-Vogtmann)

There is a stable embedding

$$\lim_{n\to\infty} \mathfrak{h}^+_{\mathrm{ab}}(V_n) \hookrightarrow \lim_{n\to\infty} \bigwedge^3 V_n \oplus \bigoplus_{r\geq 1} \mathscr{H}_r(\mathrm{Sym}(V_n)).$$

Moreover the Sp*-decomposition of* $\mathfrak{h}^+_{\mathrm{ab}}$ *is isomorphic to the* GL *decomposition of* $\bigoplus_{r\geq 0} \mathscr{H}_r(\mathrm{Sym}(V))$*.*

This allows us to interpret the construction of rational homology classes for $\mathrm{Out}(F_r)$ in terms of $\mathscr{H}_*(\mathrm{Sym}(V))$.

$$\lim_{n\to\infty} P\left[\bigwedge^* \mathscr{H}_*(\mathrm{Sym}(V_n))\right]^{\mathrm{Sp}} \to \bigoplus_{r\geq 2} H_*(\mathrm{Out}(F_r);\mathbb{k})$$

Relation to the Johnson homomorphism

The higher order Johnson homomorphism is a Lie algebra homomorphism

$$\tau\colon \mathrm{Gr}_{\mathbb{J}}(\mathrm{Mod}(g,1)) \to \mathfrak{h}^+(H_1(\Sigma_{g,1};\Bbbk))$$

where $\mathrm{Gr}_{\mathbb{J}}(\mathrm{Mod}(g,1))$ is the associate graded (tensored with $\Bbbk$) vector space associated with the Johnson filtration of the mapping class group $\mathrm{Mod}(g,1)$.

Theorem (Hain)

$\mathrm{im}\,\tau$ *is (stably) generated as a Lie algebra by degree* 1 *elements.*

The Johnson cokernel $\mathsf{C}(V)$ is defined to be $\mathfrak{h}^+(V)$ divided by the Lie algebra generated by degree 1 elements. Indeed, away from degree 1 we have a surjection $\mathsf{C}(V) \twoheadrightarrow \mathfrak{h}^+_{\mathrm{ab}}(V)$.

Theorem (Conant-Kassabov)

There is a map from the Johnson cokernel

$$\mathsf{C}(V) \to \bigoplus_{r\geq 1} \mathscr{H}_r(T(V)).$$

Stably, every GL*-representation in* $\mathscr{H}_r(T(V))$ *will appear as an* Sp *representation in* $\mathsf{C}(V)$*.*

Calculations in rank 1

1. $\mathscr{H}_1(\mathrm{Sym}(V)) \cong \bigoplus_{k\geq 0} \mathrm{Sym}^{2k+1}(V)$ (Morita trace)
2. $\mathscr{H}_1(T(V)) \cong \bigoplus_{k\geq 1} [V^{\otimes k}]_{D_{2k}}$ (Enomoto-Satoh trace)

Calculations in rank 2

Let $\mathscr{X}_{w,i}$ be the vector space of weight w classical cusp forms if i is even and the vector space of all classical modular forms of weight w if i is odd.

1. $\mathscr{H}_2(\mathrm{Sym}(V)) \cong \bigoplus_{k\geq\ell\geq 0} \mathbb{S}_{(k,\ell)}(V)\otimes\mathscr{X}_{k-\ell+2,\ell}$ (Conant-Kassabov-Vogtmann)
2. $\mathscr{H}_2(U\mathsf{L}_{(2)}) \cong \bigoplus_{k\geq\ell\geq 0} \overline{\mathbb{S}_{(k,\ell)}}(\mathsf{L}_{(2)})\otimes\mathscr{X}_{k-\ell+2,\ell}$. (Conant-Kassabov)

Here $\mathsf{L}_{(2)} \cong V\oplus\bigwedge^2 V$ is the free metabelian Lie algebra and $U\mathsf{L}_{(2)}$ its universal enveloping algebra. $\overline{\mathbb{S}_{(k,\ell)}}(\mathfrak{g})$ is the quotient of the Schur functor $\mathbb{S}_{(k,\ell)}(\mathfrak{g}) = \mathfrak{g}^{\otimes(k+\ell)}\otimes_{S_{k+\ell}}[(k,\ell)]_{S_{k+\ell}}$ by the adjoint action of $\mathfrak{g}$.

New result in rank 3

Theorem (Conant 2015)

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$$0 \to \mathrm{Sym}^{2k}(V) \to H^1(\mathrm{GL}_2(\mathbb{Z}); \mathrm{Sym}(V)^{\otimes 2} \otimes (det))_{2k} \to H^3(\mathrm{GL}_3(\mathbb{Z}); \mathrm{Sym}(V)^{\otimes 3})_{2k} \to 0$$

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3. *There is an isomorphism*

$$\mathscr{H}_3(\mathrm{Sym}(V))_e \cong \bigoplus_{k>\ell>0} [k,\ell]_{\mathrm{GL}} \otimes \mathscr{X}_{k-\ell+2,\ell+1} \oplus \bigoplus_{k>0} [k]_{\mathrm{GL}} \otimes \mathscr{S}_{k+2}$$

Now that we understand some reasons why the objects $\mathscr{H}_r(H) = H^{2r-3}(\mathrm{Out}(F_r); \overline{H^{\otimes r}})$ are important, we will take the time to show how they can be constructed and interpreted in a graph homology context.

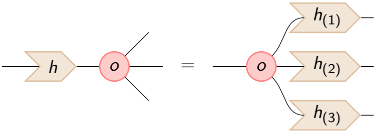
The operad $H\mathscr{O}$

Suppose H is a co-commutative Hopf algebra and $\mathscr{O}$ is an operad with unit (in the category of $\mathbb{k}$-vector spaces). We let $\mathscr{O}[n]$ denote the vector space spanned by operad elements with n inputs and one output, n being referred to as the *arity*. If $\mathscr{O}$ is cyclic, we let $\mathscr{O}((n)) = \mathscr{O}[n-1]$ as an $\mathbb{S}_n$-module.

Regard H as an operad with elements only of arity 1 and operad composition given by algebra multiplication. The antipode S turns H into a cyclic operad: the $\mathbb{S}_2$ action sends h to $S(h)$.

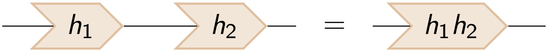
Definition

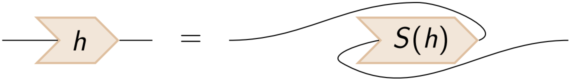
1. Let $\mathscr{O}_1$ and $\mathscr{O}_2$ be operads with unit. Define $\mathscr{O}_1 * \mathscr{O}_2$ to be the operad *freely generated by* $\mathscr{O}_1$ *and* $\mathscr{O}_2$. This is defined to be the operad consisting of trees with vertices of valence ≥ 2 labeled by elements of $\mathscr{O}_1$ or $\mathscr{O}_2$. Composing two elements of $\mathscr{O}_i$ for $i = 1, 2$ along a tree edge is considered the same element of $\mathscr{O}_1 * \mathscr{O}_2$, and the units of $\mathscr{O}_1$ and $\mathscr{O}_2$ are identified and equal to the unit of $\mathscr{O}_1 * \mathscr{O}_2$.
2. Let $H\mathscr{O}$ be the quotient of $H * \mathscr{O}$ by the relation that h commutes with an element of $\mathscr{O}$ via the comultiplication map as in the figure below.

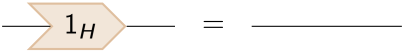


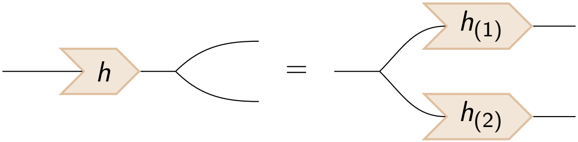
Note the use of Sweedler notation hiding the fact that the coproduct is actually a sum of pure tensors.

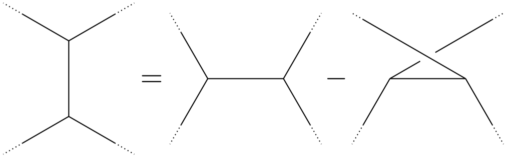
It is worthwhile to spell this construction out in a little more detail for HLie as follows. As before, we regard the H part of the operad as being a two-valent vertex with one input leaf, one output leaf and H-labeled inside. Then $H\text{Lie}((n))$ is generated by trees with vertices of valence ≤ 3, with an ordering of the edges at each bivalent vertex, a cyclic ordering of the edges at the trivalent vertices, and the bivalent vertexes are labeled with elements of the Hopf algebra H. These trees satisfy the following relations:

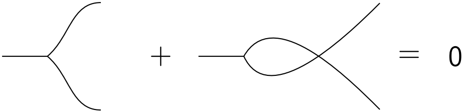
1. (Multiplication) 

2. (Antipode) 

3. (Removal of the Identity) 

4. (Comultiplication) 

5. (IHX) 

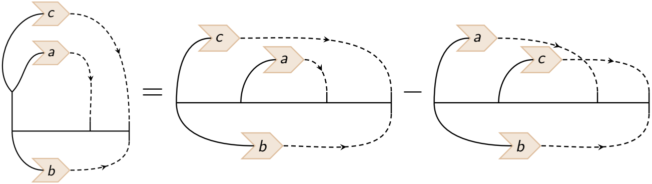
6. (AntiSymmetry) 

Graph homology

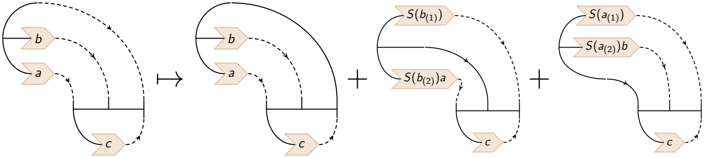
One can define a graph complex $\mathscr{G}_{\mathscr{O}}$ for any cyclic operad $\mathscr{O}$ by putting elements of $\mathscr{O}((|v|))$ at each vertex v of a graph and identifying the i/o slots with the adjacent edges. In this definition, the graph may have bivalent vertices but no univalent or isolated vertices, since the operad $\mathscr{O}$ is assumed not to have anything in arity -1 and 0.

These complexes are graded by the number of vertices of the underlying graph and the boundary operator is induced by contracting edges of the underlying graph. Let $\mathscr{G}_{\mathscr{O}}^{(n)}$ be the subcomplex spanned by $\mathscr{O}$-colored connected graphs of rank n.

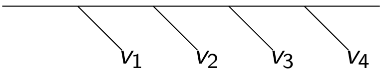
An IHX relation in $\mathscr{G}_{H\mathrm{Lie},1}$. Here $a, b, c \in H$.



A boundary from $\mathscr{G}_{H\mathrm{Lie},2} \to \mathscr{G}_{H\mathrm{Lie},1}$.



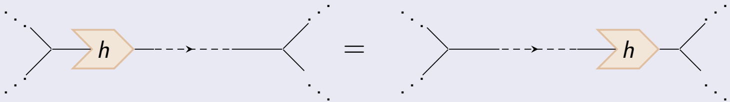
Remark: The Hopf algebra elements $h \in H$ encode the "hairs" of hairy graph homology. For example the product $v_1 v_2 \cdots v_k \in T(V)$ represents k hairs in a row, labeled by $v_1, \ldots, v_k$. When $H = \mathrm{Sym}(V)$, these hairs commute, which is what happens for hairy Lie graph homology.



$\rightsquigarrow \quad v_1 v_2 v_3 v_4 \in H$

Definition

Let $\overline{\mathscr{G}_{H\mathrm{Lie}}}$ denote the quotient of the graph complex for HLie where the elements in H are allowed to slide through the edges, i.e., the following graphs in $\mathscr{G}_{H\mathrm{Lie}}$ are equivalent in $\overline{\mathscr{G}_{H\mathrm{Lie}}}$.



It is clear that the quotient map $\mathscr{G}_{H\mathrm{Lie}} \to \overline{\mathscr{G}_{H\mathrm{Lie}}}$ preserves the differential and induces a map between the homologies.

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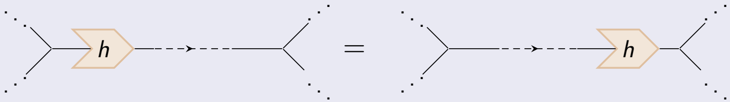

It is clear that the quotient map $\mathscr{G}_{H\mathrm{Lie}} \to \overline{\mathscr{G}_{H\mathrm{Lie}}}$ preserves the differential and induces a map between the homologies.

Theorem (Conant-Kassabov)

For $n \geq 2$ we have $H_k(\overline{\mathscr{G}^{(n)}_{H\mathrm{Lie}}}) = H^{2n-2-k}(\mathrm{Out}(F_n); \overline{H^{\otimes n}})$.

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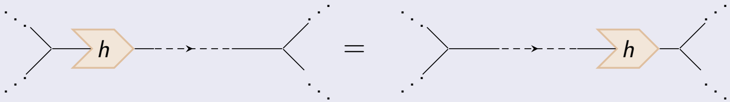

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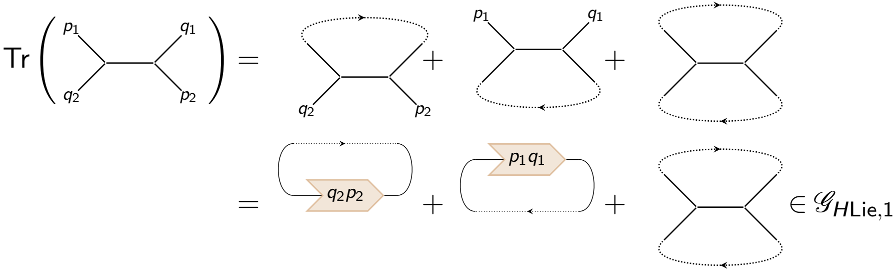
- Then $\mathscr{H}_n(H) = H_1(\overline{\mathscr{G}^{(n)}_{H\text{Lie}}})$.

The trace map

Define a map $\mathrm{Tr}\colon \mathfrak{h}(V) \to \mathscr{G}_{H\mathrm{Lie},1}$ by summing over adding external edges in all ways, contracting by the labeling coefficients.



- When $H = \mathrm{Sym}(V)$, one can show that $\mathrm{Tr}([\mathfrak{h}^+, \mathfrak{h}^+]) \subset \partial\mathscr{G}_{H\mathrm{Lie},2}$, so gives a map

$$\mathfrak{h}^+_{\mathrm{ab}} \to H_1(\mathscr{G}_{H\mathrm{Lie},\cdot}) \cong \mathscr{H}_*(\mathrm{Sym}(V)).$$

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- When $H = T(V)$, one can show that $\mathrm{Tr}(\langle \mathfrak{h}_1(V) \rangle) \subset \partial\mathscr{G}_{H\mathrm{Lie},2}$, so gives a map $\mathrm{C}(V) \to H_1(\mathscr{G}_{H\mathrm{Lie},\cdot}) \cong \mathscr{H}_*(T(V))$.

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- In fact, there is an even stronger invariant of the Johnson cokernel. Let $\mathscr{S}_2 \subset \mathscr{G}^{(n)}_{H\mathrm{Lie},2}$ be spanned by graphs where one of the two HLie elements is actually an element of $\mathrm{Lie}((3)) \subset H\mathrm{Lie}((3))$, i.e. it is a tripod where all three i/o slots are joined to graph edges. We define $\Omega_n(H) = \overline{\mathscr{G}^{(n)}_{H\mathrm{Lie},1}} / \partial \mathscr{S}_2$.

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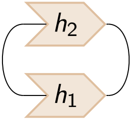
- When $H = T(V)$, one can show that $\mathrm{Tr}(\langle \mathfrak{h}_1(V) \rangle) \subset \partial \mathscr{S}_2$, so gives a map $\mathrm{C}(V) \to \Omega_*(T(V))$

Rank 1

Theorem

There is an isomorphism $\mathscr{H}_1(H) \cong (\mathrm{Id} - S)(H/[H,H])$. *In particular*

1. $\mathscr{H}_1(\mathsf{Sym}(V)) \cong \bigoplus_{k\geq 0} \mathsf{Sym}^{2k+1}(V)$
2. $\mathscr{H}_1(T(V)) \cong \bigoplus_{k\geq 1} [V^{\otimes k}]_{D_{2k}}$

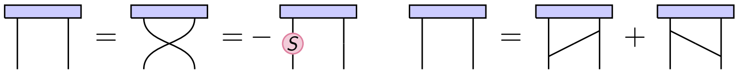


Rank 2

Theorem

$\mathscr{H}_2(H)$ is the quotient of $\overline{H^{\otimes 2}}$ by

1. $(\mathrm{Id}-\sigma_{12})\overline{H^{\otimes 2}}$ *I.e.* $a\otimes b = b\otimes a$
2. $(\mathrm{Id}+\varepsilon)\overline{H^{\otimes 2}}$ *I.e.* $a\otimes b = -S(a)\otimes b$
3. $(\mathrm{Id}+\gamma+\gamma^2)\overline{H^{\otimes 2}}$ *I.e.* $a\otimes b + S(b)a_{(1)}\otimes a_{(2)} + b_{(1)}\otimes S(a)b_{(2)} = 0$, *using modified Sweedler notation.*



Rank 3

In the following, let $\sigma_{ij}\colon H^{\otimes 3} \to H^{\otimes 3}$ transpose the ith and jth factors. Let $E\colon H^{\otimes 3} \to H^{\otimes 3}$ be defined by $E = (\mathrm{Id} \otimes m \otimes \mathrm{Id}) \circ (\Delta \otimes \mathrm{Id} \otimes \mathrm{Id})$, and similarly $F = (m \otimes \mathrm{Id} \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes \Delta \otimes \mathrm{Id})$.

Theorem

$\mathscr{H}_3(H) \cong H^3(\mathrm{Out}(F_3); \overline{H^{\otimes 3}})$ *is isomorphic to the quotient of* $\overline{H^{\otimes 3}}$ *by the images of the following operators (operating on the left):*

1. $\mathrm{Id} + \sigma_{12}(S \otimes S \otimes S)$
2. $\mathrm{Id} + \sigma_{13} - \sigma_{23}\sigma_{12} - \sigma_{12}$
3. $(\mathrm{Id} + S) \otimes \mathrm{Id} \otimes \mathrm{Id} + \sigma_{23}(\mathrm{Id} + S) \otimes \mathrm{Id} \otimes \mathrm{Id}$
4. $\mathrm{Id} + \sigma_{23}E + \sigma_{13}F - \sigma_{12} - \sigma_{13}F\sigma_{12} - \sigma_{23}E\sigma_{12}$
5. $(\mathrm{Id} + \sigma_{23}E + \sigma_{13}F)((\mathrm{Id} + S) \otimes \mathrm{Id} \otimes \mathrm{Id})$
6. $\sigma_{23}E(S \otimes \mathrm{Id} \otimes \mathrm{Id}) + \sigma_{13}F\sigma_{23}(S \otimes \mathrm{Id} \otimes \mathrm{Id}) + \sigma_{23}E + \sigma_{13}F$

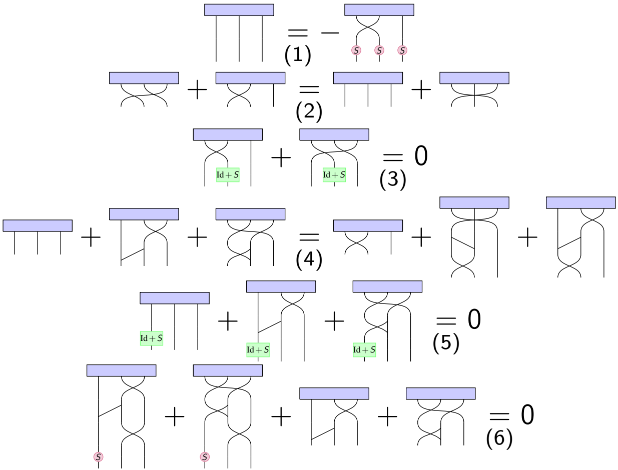


Figure: $\mathscr{H}_3(H)$ is isomorphic to $\overline{H^{\otimes 3}}$ modulo the above relations.

Special case: $H = \mathrm{Sym}(V)$

Next we consider the specific case of $H = \mathrm{Sym}(V)$. Decompose $\mathrm{Sym}(V)^{\otimes 3} \cong [\mathrm{Sym}(V)^{\otimes 3}]_e \oplus [\mathrm{Sym}(V)^{\otimes 3}]_o$ into even and odd degree pieces respectively. The even degree case is particularly simple:

Theorem

$\mathscr{H}_3(\mathrm{Sym}(V))_e \cong H^3(\mathrm{Out}(F_3); [\mathrm{Sym}(V)^{\otimes 3}]_e)$ is isomorphic to the quotient of $[\mathrm{Sym}(V)^{\otimes 3}]_e$ by the images of the operators below.

1. $\mathrm{Id} + \sigma_{12}$
2. $\mathrm{Id} + \sigma_{23}$
3. $\mathrm{Id} + S \otimes \mathrm{Id} \otimes \mathrm{Id}$
4. $E + F - \mathrm{Id}$

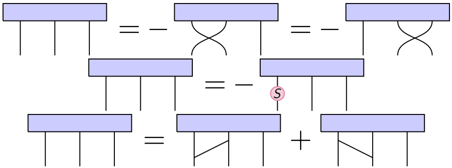


Figure: $\mathscr{H}_3(\mathrm{Sym}(V))_e$ is presented by $[\mathrm{Sym}(V)^{\otimes 3}]_e$ modulo these relations.

Proposition

$$H^3(\mathrm{Out}(F_3);[\mathrm{Sym}(V)^{\otimes 3}]_e) \cong H^3(\mathrm{GL}_3(\mathbb{Z});[\mathrm{Sym}(V)^{\otimes 3}]_e)$$

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Theorem

For $k \geq 1$, there is an exact sequence

$$0 \to \mathrm{Sym}^{2k}(V) \to H^1(\mathrm{GL}_2(\mathbb{Z});\mathrm{Sym}(V)^{\otimes 2} \otimes (det))_{2k} \to H^3(\mathrm{GL}_3(\mathbb{Z});\mathrm{Sym}(V)^{\otimes 3})_{2k} \to 0$$

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- Proof: The map $H^1(\mathrm{GL}_2(\mathbb{Z}); \mathrm{Sym}(V)^{\otimes 2} \otimes (det))_{2k} \to H^3(\mathrm{GL}_3(\mathbb{Z}); \mathrm{Sym}(V)^{\otimes 3})_{2k}$ is obvious from the presentations and it is straightforward to calculate the kernel. Surjectivity follows from classical work of Borel and Wallach.

Corollary

$$\mathscr{H}_3(\mathsf{Sym}(V))_e \cong \bigoplus_{k>\ell>0} [k,\ell]_{\mathsf{GL}} \otimes \mathscr{X}_{k-\ell+2,\ell+1} \oplus \bigoplus_{k>0} [k]_{\mathsf{GL}} \otimes \mathscr{S}_{k+2}$$

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- Proof: Stems from the Eichler-Shimura isomorphism $H^1(\mathrm{SL}_2(\mathbb{Z}); \Bbbk[x,y]_{2k}) \cong \mathscr{S}_{2k+2} \oplus \mathscr{M}_{2k+2}$.

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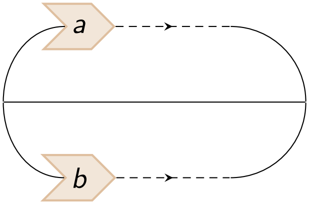
- This yields rank 3 classes of the abelianization of $\mathfrak{h}^+$.

A proof of the presentation of $\mathscr{H}_2(H)$.

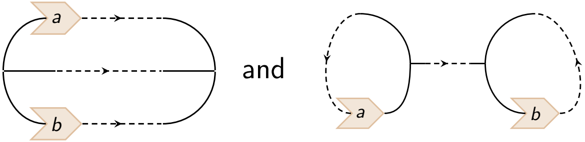
$\mathscr{G}^{(2)}_{H\mathrm{Lie},1}$ is generated by graphs of the form



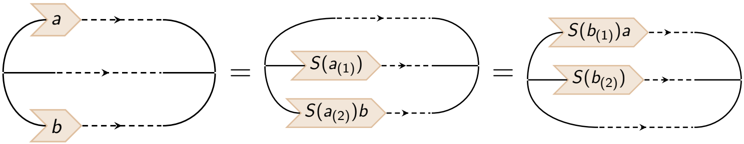
which we identify with $a \otimes b \in \overline{H^{\otimes 2}}$. (The element $a \otimes b$ is only well defined up to conjugation.) The other type of graph, the eyeglass graph, is a linear combination of two of these via an IHX relation.

The various symmetries of the graph lead to the relations $a \otimes b = b \otimes a = S(a) \otimes S(b)$.

Now $\partial(\mathscr{G}^{(2)}_{H\mathrm{Lie},2})$ is generated by the boundaries of the following two types of graphs:



The boundary of the first graph has three terms corresponding to contracting along each of the three dashed edges. Contracting along the middle edge gives $a \otimes b$. To contract along the other two edges, we use the fact that



So the boundary becomes
$a \otimes b + S(a_{(1)}) \otimes S(a_{(2)})b + S(b_{(1)})a \otimes S(b_{(2)})$.

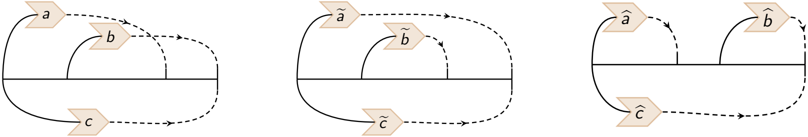
The boundary of the second graph has only one term, which is equal by an IHX relation to $-a \otimes S(b) - a \otimes b$. This gives the relation $a \otimes b = -S(a) \otimes b$, and together with symmetry derived above, its consequence $a \otimes b = S(a) \otimes S(b)$.

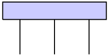
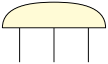
Thus $\mathscr{H}_2(H)$ is the quotient of $\overline{H^{\otimes 2}}$ by the relations

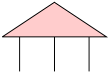
1. $a \otimes b + S(a_{(1)}) \otimes S(a_{(2)})b + S(b_{(1)})a \otimes S(b_{(2)}) = 0$
2. $a \otimes b = S(a) \otimes S(b)$
3. $a \otimes b = b \otimes a = S(a) \otimes S(b)$

Presentation of $\mathscr{H}_3(H)$

the three generators of $\overline{\mathscr{G}^{(3)}_{H\mathrm{Lie},1}}$ are given below.



We identify them with elements of $\overline{H^{\otimes 3}}$. So the first generator corresponds to $a \otimes b \otimes c \in H^{\otimes 3}$. In order to distinquish the three graphs, we consider three copies of $\overline{H^{\otimes 3}}$, with elements of the second copy denoted $\widetilde{a} \otimes \widetilde{b} \otimes \widetilde{c}$ and elements of the third copy denoted $\widehat{a} \otimes \widehat{b} \otimes \widehat{c}$. Depicting these three types of tensors pictorially, we use the symbols , , and

.

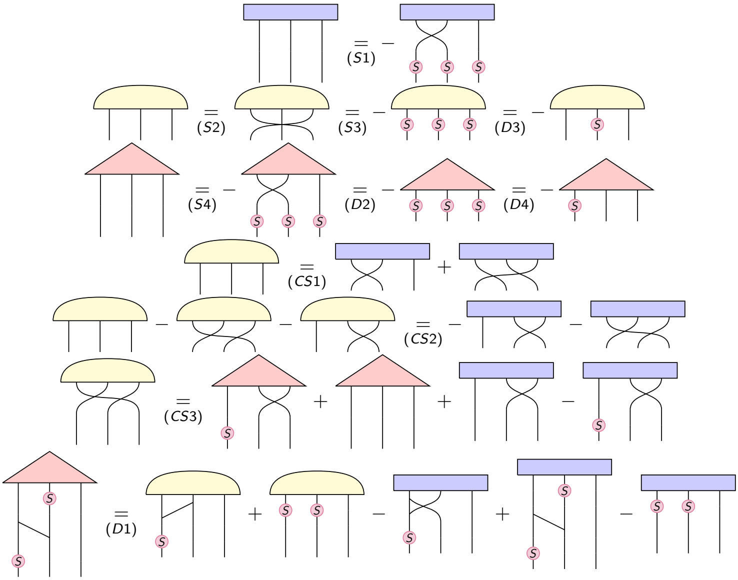


Figure: Rank 3 relations in pictorial format

degree	$\mathscr{H}_2(\mathrm{Sym}(V))$	$\mathscr{H}_2(T(V))$	$\Omega_2(\mathrm{Sym}(V))$	$\Omega_2(T(V))$
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	[31]	[31]	$[4] \oplus [31]$	$[4] \oplus [31]$
5	0	$[31^2] \oplus [2^2 1] \oplus [21^3]$	0	$2[31^2] \oplus [2^2 1] \oplus [21^3]$
6	[51]	$[1^6] \oplus [51] \oplus [21^4]$ $\oplus [42] \oplus 2[2^2 1^2] \oplus [3^2]$ $\oplus [2^3] \oplus 2[321]$	$[6] \oplus 2[51]$ $\oplus [42]$	$[1^6] \oplus 2[51] \oplus 2[21^4]$ $\oplus 3[42] \oplus 2[2^2 1^2] \oplus [3^2]$ $\oplus 2[2^3] \oplus 3[321] \oplus [6]$
7	0	?	0	?
8	$[71] \oplus [53]$	?	$2[8] \oplus 2[71]$ $\oplus 2[62] \oplus [53]$	?

Figure: Rank 2 Computations

degree	$\mathscr{H}_3(\mathrm{Sym}(V))$	$\mathscr{H}_3(T(V))$	$\Omega_3(\mathrm{Sym}(V))$	$\Omega_3(T(V))$
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	$[21]$	$[21]$	$[21]$	$[21]$
4	0	$[1^4]$	$[4] \oplus [31]$	$[4] \oplus [31] \oplus [21^2] \oplus [1^4]$
5	$[41] \oplus [32]$	$[41] \oplus [32]$	$[31^2] \oplus [2^2 1] \oplus 3[41] \oplus 3[32] \oplus 2[5]$	$2[5] \oplus 3[41] \oplus [21^3] \oplus 3[31^2] \oplus 4[32] \oplus 3[2^2 1]$
6	$[42]$	?	$[6] \oplus 2[51] \oplus [42]$	?
7	$[7] \oplus 2[61] \oplus 2[52] \oplus [51^2] \oplus [43] \oplus [421] \oplus [3^2 1]$	?	$5[7] \oplus 8[61] \oplus 8[52] \oplus 3[51^2] \oplus 5[43] \oplus 4[421] \oplus [3^2 1] \oplus [32^2]$	?
8	$[62]$	?	$2[8] \oplus 2[71] \oplus 2[62] \oplus [53]$	?

Figure: Rank 3 Computations