RYSZARD 4

$N \subset M_0 \subset M_1 \subset \cdots$

$N' \cap N \subset N' \cap M_0 \subset N' \cap M_1 \subset \cdots$

N–N BIMODULES (IRREDUCIBLES)

RESTRICTION

$\otimes_N L^2(M)_M$

$_M L^2(M) \otimes_M -$

N–M-BIMODULES

M–N-BIMODULES

M–M-BIMODULES

PRINCIPAL GRAPH

DUAL PRINCIPAL GRAPH

OPPOSITE PRINCIPAL GRAPH

PRINCIPAL GRAPH: FINITE GRAPH BY FINITE DEPTH ASSUMPTION

FACT: INDEX OF $N \subset M$

= NORM OF PRINCIPAL GRAPH

(= OPERATOR NORM OF INCIDENCE MATRIX OF THE GRAPH)

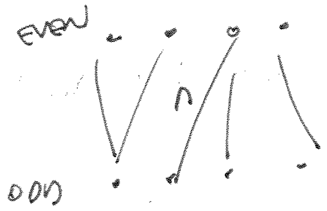
= BIGGEST EIGENVALUE

COROLLARY: $[M:N]$

$\uparrow$

$\{4\cos^2\frac{\pi}{n}\} \cup [4,\infty)$

$n \geq 3$



EVEN DOTS

ODD DOTS

$\begin{pmatrix} & & \\ & n & \\ & & \end{pmatrix}$

MULTIPLICITY

(BECAUSE NORM OF A ~~[illegible]~~ MATRIX WITH POSITIVE INTEGER COEFF)

$$M_{k+1} = \langle M_k, e_{k+1} \rangle$$

THE PROJECTIONS $e_1 \quad e_2 \quad e_3 \quad \cdots\cdots$

SATISFIE THAT $e_i e_j = e_j e_i$ if $|j-i|>1$

AND $e_4 e_3 e_4 = \frac{1}{[M:N]} e_4$, $e_3 e_4 e_3 = \frac{1}{[M:N]} e_3$

$\leadsto$ FORM A TEMPORI-LIEB ALG.

CAN ALSO CONSIDER THE RELATIVE COMMUTANTS

$$\begin{tikzcd}
\cdots M_k' \cap M_n \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup" rotate=-90] & M_k' \cap M_{n+1} \arrow[l, bend right] \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup" rotate=-90] & \cdots \\
\cdots M_{k+1}' \cap M_n \arrow[r, phantom, "\subset"] \arrow[u] & M_{k+1}' \cap M_{n+1} \arrow[l, bend right] \arrow[r, phantom, "\subset"] \arrow[u] & \cdots
\end{tikzcd}$$

MORE GENERALLY.

$\leadsto$ DIAGRAM OF FINITE DIM ALGEBRAS.

AND THE DIAGRAM OF CONDITIONAL EXPECTATIONS COMMUTES.

FROM SUCH COMMUTING SQUARES CAN CONSTRUCT SUBFACTORS.

$$\begin{tikzcd}
A_2 \arrow[r, shift left] \arrow[d, shift right] & A_3 \arrow[l, shift left] \arrow[d, shift left] \\
A_0 \arrow[r, shift left] \arrow[u, shift right] & A_1 \arrow[l, shift left] \arrow[u, shift left]
\end{tikzcd}$$

(GETTING SUCH SQUARES IS NOT EASY)

EXAMPLE OF $N \rtimes G$: (G FINITE)

$$\mathbb{C} \subset C(G) \subset \underset{B(\ell^2(G))}{\underset{\shortparallel}{C(G)\rtimes G}} \subset \underset{C(G)\otimes B(\ell^2(G))}{\underset{\wr\shortparallel}{(C(G)\rtimes G)\rtimes \hat{G}}} \subset \cdots$$

$$\rightsquigarrow \quad \begin{array}{ccc} C^*(G) & \hookrightarrow & B(\ell^2(G)) \\ \downarrow\uparrow & & \uparrow\downarrow \\ \mathbb{C} & \rightleftarrows & C(G) \end{array}$$

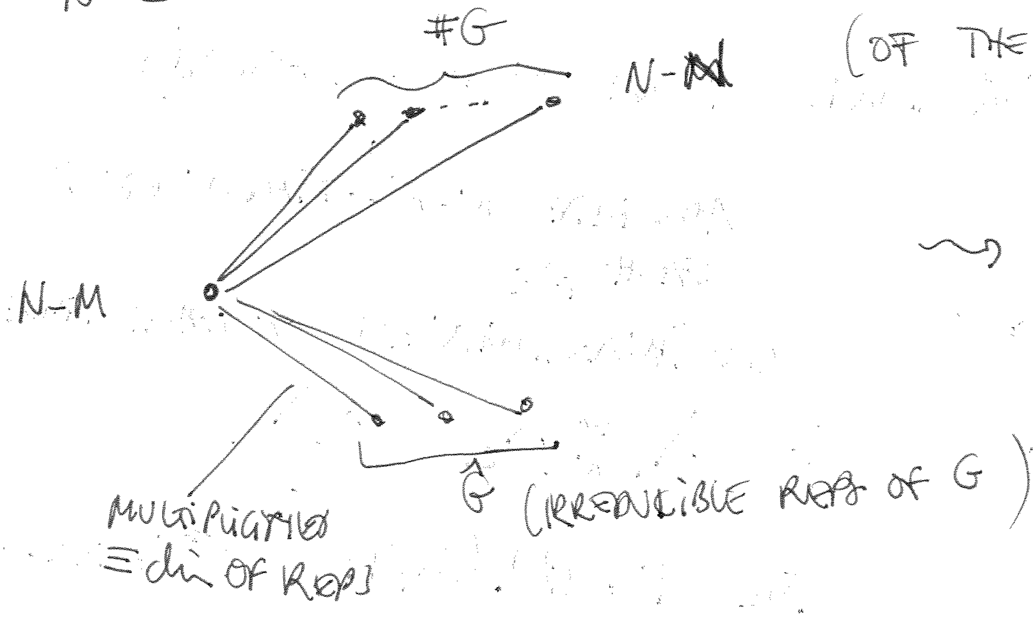
COMMUTING SQUARE
i.e. WITH THE EXPECTATIONS

convolution

$$\begin{array}{ccc} & \sum (\varphi * \cdot)(\psi \cdot\cdot) & \\ \swarrow & & \searrow \\ \int \psi \cdot (\varphi * \cdot) & & \varphi(e)(\psi \cdot\cdot) \\ \searrow & & \swarrow \\ & \int \psi \cdot \varphi(e) & \end{array}$$

$$N \subset N\rtimes G \subset N\rtimes G\rtimes\hat{G} \subset N\rtimes G\rtimes\hat{G}\rtimes G \subset \cdots$$

(OF THE FORM N_g)



$$\rightsquigarrow \quad (\#G)^2 = \sum_{\pi} (\dim \pi)^2$$

① N-N-BIMODULES

$\rightsquigarrow$ FINITE TENSOR CATEGORY:

$$X \otimes Y = \bigoplus_i X_i \otimes m_i$$

<u>BRAIDING</u>: IN GENERAL, ${}_N X_N \otimes {}_N Y_N \neq {}_N Y_N \otimes {}_N X_N$, BUT...

$N \subset M \rightsquigarrow$ REPLACE BY ASYMPTOTIC INCLUSION

$$M_0 \cup (M_0' \cap M_\infty) \subset M_\infty$$

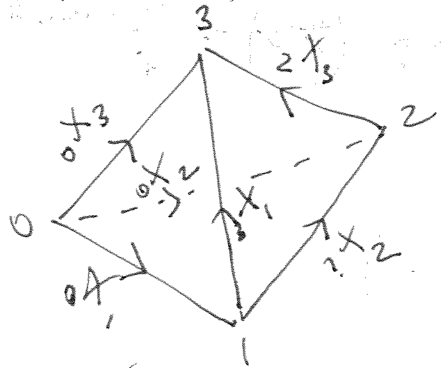
ALSO FINITE INDEX

ANALOGUE (CORRESPONDS TO THE DRINFELD CENTER AT THE DOUBLE LEVEL OF TENSOR CATEGORIES)

$$M_\infty = \overline{\bigcup_i M_i}$$

NOW N-N-BIMODULES, M-M-BIMODULES } BRAIDED TENSOR CATEGORIES [OCNEANU]

TURAEV-VIRO INVARIANTS FROM SUCH CATEGORIES



ASSOCIATE N-N-BIMODULES TO EACH ${}_iX_j$

AND INTERTWINERS ON EACH FACE

eg ${}_3X_1 \otimes_N {}_1X_2 \longrightarrow {}_3X_2$

FOR $T \in \mathrm{End}(X,Y)$, $T^*T \in \mathrm{End}(X)$

$$\|T\|_2^2 = \tau(T^*T)$$

GET 4 INTERWINNERS THAT CAN BE COMPOSED TO GIVE A SELF INTERTWINER OF A SINGLE BIMODULE $\rightsquigarrow$ INTERTWINERS FORM HILBERT SPACES

$\longrightarrow$ A <u>NUMBER</u> $\in \mathbb{C}$ (6-j-SYMBOL)

$$\rightsquigarrow \sum_{\text{DECORATIONS}} \prod (\text{6-j-SYMBOLS}) = \text{3-MFD INVARIANT}$$

NEED THE CATEGORY OF BIMODULES TO HAVE
CERTAIN PROPERTIES FOR THIS TO WORK...