RYSZARD · II

G ACTING ON N (II_1-FACTOR) BY OUTER AUT

$$N \subset N \rtimes G \subset N \rtimes G \rtimes \hat{G} \subset \cdots$$

$\xleftarrow{E_0}$

$$n_e \leftarrow\!\shortmid \sum n_g \cdot g$$

N - BIMODULE MAP

$M_0 = N \rtimes G$ ALSO II_1 - FACTOR

$$N \underset{\xleftarrow{E_0}}{\subset} M_0$$

E_0 = CONDITIONAL EXPECTATION FROM HIROSHI'S LECTURE 2. (DEFINED EXPLICITLY ABOVE)

STANDARD FORM FOR $M_0 \xrightarrow{\pi_e} B(L^2(M_0))$

$N \longrightarrow B(L^2(M_0)) \longleftarrow N'$ = COMMUTANT IN $L^2(M_0)$

$\exists\, J : L^2(M_0) \longrightarrow L^2(M_0)$ s.t. $J M_0 J = M_0'$

$\rightsquigarrow$ CAN USE J TO WRITE A RIGHT REP OF M_0:

$$\pi_e : M_0 \longrightarrow B(L^2(M_0))$$
$$n \longmapsto J\,\pi_e(n^*)\,J$$

$\rightsquigarrow$ $L^2(M_0)$ IS AN M_0 - BIMODULE.

[RECALL: $L^2(M_0, \tau)$ HAS $\|m\|_\tau^2 = \tau(m^*m)$
τ TRACE
AND $J : m \mapsto m^*$]

$$N \subset M_0 \subset JN'J =: M_1$$

AS
$$\begin{array}{ccccc} M_0 & & & & M_0' \\ \cup & & & & \cap \\ N & \longrightarrow & B(L^2(M_0)) & \longleftarrow & N' \end{array}$$
AND $JM_0'J = M_0$

CLAIM: IN OUR EXAMPLE, THIS IS THE SAME M_1

CAN CONTINUE ⇝ PRODUCE $M_2, \ldots$

OTHER EX: Γ = DISCRETE ∞-GROUP (COUNTABLE)

$$\begin{array}{ccccccc} N & \subset & N \rtimes \Gamma & \subset & (N \rtimes \Gamma) \rtimes \hat{\Gamma} & \subset & \ldots \\ \mathrm{II}_1\text{-FACTOR} & & \mathrm{II}_1\text{-FACTOR} & & \mathrm{II}_\infty\text{-FACTOR} & & \end{array}$$
(SEMI-FINITE TRACE)

⇝ NEED SOME CONDITION IN THE GENERAL CONSTRUCTION ABOVE TO ENSURE M_1 IS II_1.

$N \longrightarrow B(H)$ CONTINUOUS WRT WTO
(NORMAL REPRESENTATION)

vN ALG: $N = (N_*)^*$ (AS A BANACH SPACE)

$B(H) = (\mathcal{L}^1(H))^*$

HENCE

$$H \simeq L^2(N)^n \cdot e \qquad e \in M_n(N)$$

CAN WRITE TRACE $\tau(e) = \sum_i \tau(e_{ii}) := \dim_N H$

CONDITION: "FINITE INDEX": $\dim_N L^2(M_0) < \infty$
$N \subset M_0$

EX: $N \subset N \rtimes G$ $\qquad L^2(N \rtimes G) \simeq L^2(N) \otimes \mathbb{C}^{\#G}$

$$\sum n_g g \longrightarrow \{n_g\}_{g \in G}$$

$$H = L^2(N)^{\#G} \cdot \mathrm{Id}$$

$$\rightsquigarrow \dim_N L^2(\cancel{M}N \rtimes G) = \#G$$

CLAIM: IF $N \subset M_0$ IS FINITE INDEX, THEN
(N, M_0 II_1-FACTORS)
$N \subset M_0 \subset M_1 \subset M_2 \subset \dots$ CONSTRUCTED AS ABOVE IS A SEQUENCE OF II_1-FACTORS.

WHERE WE ARE GOING:

$(M\text{-}M)$-BIMODULES

INDUCE $\nearrow$ $\quad \swarrow$ RESTRIC

$(M\text{-}N)$-BIMODULES

$${}_M L^2(M)_M \quad \dots\dots$$

$${}_M L^2(M)_N$$

FINITE DEPTH: ONLY FINITELY MANY $\neq$ BIMODULES OCCUR.

$\rightsquigarrow$ GET A GRAPH OF BIMODULES

$\rightsquigarrow$ "FUSION SYSTEM" $\rightsquigarrow$ INVARIANTS OF 3-MFDS.

EXAMPLE OF $N \subset M$: N VN ALG ~~##~~ (II_1-FACTOR),

$H \subset G$, $G \overset{\text{ACTS}}{\curvearrowright} N \rightsquigarrow N^H \supset N^G$

$M_1 = J N' J = \mathrm{End}(L^2(M_0)_N)$

AS RIGHT N-MODULE (VIA JNJ)

$M_2 = \mathrm{End}(L^2(M_1)_{M_0})$

$N \underset{E_0}{\subseteq} M_0 \subseteq M_1 \subset \dots$

$L^2(M_0) \underset{N}{\otimes} L^2(M_0)$ DEFINE $\langle m_1 \otimes m_2 \mid m_1' \otimes m_2' \rangle$ (SCALAR PRODUCT)

$= \langle E_0((m_1')^* m_1) m_2 \mid m_2' \rangle$

COMPLETE $\longrightarrow$ HILBERT SPACE

$\rightarrow$ HAVE A TENSOR PRODUCT OF N-MODULES.

HERE $L^2(M_1) = L^2(M_0) \underset{N}{\otimes} L^2(M_0)$.

$m \underset{N}{\otimes} m' \in M_0 \underset{N}{\otimes} M_0$ (AS VECTOR SPACES) ACTS ON $L^2(M_0)$:

$(m \otimes_N m')(\underset{\in M_0}{x}) = m\, E_0(m' x) \implies$ NEED TO DEFINE

$(m_1 \underset{N}{\otimes} m_1')(m_2 \otimes m_2') = m_1 E(m_1' m_2) \otimes m_2'$

$$L^2(M_1) = L^2(M_0) \otimes_N L^2(M_0)$$

FINITE INDEX

$\Rightarrow M_1 = M_0 \otimes_N M_0$, $m \otimes m' : L^2(M_0) \to L^2(M_0)$

WITH MULTIPLICATION AS ABOVE.

IN PARTICULAR, $\mathbb{1}_{L^2(M_0)} \in \mathrm{End}(L^2(M_0)_N) = M_1$

$$\| \qquad \sum_i m_i^* \otimes m_i \quad \text{(FINITE SUM)}$$

$\Rightarrow m \in L^2(M_0)$, $m = \sum_i m_i^* E(m_i m)$ $\rightsquigarrow$ m_i^* GIVE A BASIS AS N-MODULE.

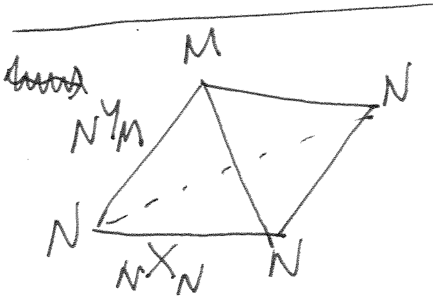
① $e = 1 \otimes_N 1$ — "PROJECTION"

② $M_1 = \langle M_0, e \rangle$ $\qquad M_0 \hookrightarrow M_1$, $m \longmapsto m \otimes 1$

③ $e(m \otimes 1)e = (1 \otimes 1)(m \otimes 1)(1 \otimes 1)$

$$= (1 \otimes 1)(m \otimes 1) = 1 \otimes E(m) = (1 \otimes 1)E(m)$$

$\Rightarrow eme = E(m)e$.



WILL DECORATE TETRAHEDRA BY ALGEBRAS, BIMODULES, ...

$\Rightarrow$ NUMBER $\rightsquigarrow$ INVARIANTS