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HIROSHI 2

(A) STANDARD FORM OF A II_1-FACTOR

LET M BE A VON N ALG WITH τ FAITHFUL NORMAL TRACIAL STATE. $(\tau(1)=1)$

RECALL

GNS CONSTRUCTION FOR τ : $(L^2(M,\tau), \xi_\tau, \pi)$

$L^2(M) = \overline{M}^{\langle , \rangle_\tau}$ $\quad \langle b, a\rangle_\tau = \tau(b^*a)$, $\quad a, b \in M$

$N_\tau = \{x \in M \mid \tau(x^*x) = 0\} = \{0\}$

$M \ni a \longmapsto \hat{a} \in L^2(M)$, $\quad \xi_\tau = \hat{1}$

$\pi(a)\hat{b} = \widehat{ab}$. CHECK π IS FAITHFUL.

SO IDENTIFY a AND $\pi(a)$.

$M \curvearrowright L^2(M)$.

• $J : \hat{M} \longrightarrow \hat{M} \overset{\text{DENSE}}{\subset} L^2(M)$

$\quad \hat{a} \longmapsto \hat{a^*}$

$\|J\hat{a}\|_2^2 = \|\hat{a^*}\|_2^2 = \tau(aa^*) = \tau(a^*a) = \|\hat{a}\|_2^2$

$\rightarrow$ J EXTENDS TO A CONJUGATE-LINEAR ISOMETRY $L^2(M) \circlearrowleft$

$J^2 = id_{L^2(M)}$.

LEMMA: $JMJ \subset M'$

PF: LET $a, b, c \in M$ $\qquad JaJb\hat{c} = JaJ\widehat{bc} = Ja\widehat{c^*b^*} = \widehat{bca^*}$

$$= b\widehat{ca^*} = bJ\widehat{ac^*} = bJaJ\hat{c}$$

$$\hat{M} \subset L^2(M) \text{ DENSE}$$

$$\Longrightarrow JaJb = bJaJ \ \forall b \Rightarrow JaJ \in M'$$

THM: $JMJ = M'$

PF: STEP 1: $x' \in M'$, $Jx'\xi_\tau \overset{?}{=} (x')^*\xi_\tau$

LET $a \in M$. $\langle Jx'\xi_\tau, \hat{a}\rangle = \langle J\hat{a}, x'\xi_\tau\rangle$

$$= \langle \widehat{a^*}, x'\xi_\tau\rangle$$

$$= \langle (x')^* a^* \xi_\tau, \xi_\tau\rangle$$

$$= \langle (x')^*\xi_\tau, \underbrace{a\xi_\tau}_{=\hat{a}}\rangle$$

AGAIN BY DENSITY OF $\hat{M}$, $Jx'\xi_\tau = (x')^*\xi_\tau$.

STEP 2: $M'\xi_\tau = \{a'\xi_\tau, a' \in M'\} \overset{\text{DENSE}}{\subset} L^2(M)$

LET $E := \overline{M'\xi_\tau}$ $\qquad P =$ PROJ OF $L^2(M)$ ONTO E.

WANT TO SHOW $P = 1$.

1) $P \in M$ $\qquad \bullet$ E IS M'-INVARIANT

$\bullet$ $E^\perp$ IS ______

$\Rightarrow P$ COMMUTES WITH M'

$\Rightarrow P \in M'' = M$.

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SINCE $1 \in M'$, $\xi_\tau \in E$. SO $p\xi_\tau = \xi_\tau$.

$\Leftrightarrow 0 = \|(p-1)\xi_\tau\|_\tau^2 = \tau((1-p)^*(1-p)) \Rightarrow p = 1$ ✓

STEP 3 LET $x' \in M'$, $y', z' \in M'$

$$J x' J y' (z' \xi_\tau) = J x' (z')^* (y')^* \xi_\tau = y' z' (x')^* \xi_\tau$$

$$\|$$

$$y' J x' J (z' \xi_\tau) = y' J x' (z')^* \xi_\tau = y' z' (x')^* \xi_\tau$$

BY STEP 2, $M'\xi_\tau$ DENSE $\Rightarrow Jx'Jy' = y'Jx'J$

$\Rightarrow Jx'J \in (M')' = M$

$\Rightarrow$ $M \curvearrowright L^2(M)$ BY LEFT ACTION.

BY $JMJ = M'$, $L^2(M) \circlearrowleft M$ BY $\xi \cdot x := Jx^*J\xi$

$\xi \in L^2(M), x \in M$

③ CONDITIONAL EXPECTATION FOR $N \subset M$.

THM: LET $1 \in N \subset M$ BE AN INCLUSION OF VN ALGS, τ: f n. TRACE ON M. THEN $\exists!\ E_N : M \to N$ LINEAR, POSITIVE, S.T. (1) $E_N(axb) = a E_N(x) b \quad a, b \in N, x \in M$

(2) $\tau(E_N(x)) = \tau(x), \quad x \in M$.

PF: $M \curvearrowright L^2(M) \ni \xi_\tau \qquad \mathcal{H} = \overline{N \xi_\tau} \subset L^2(M)$.

$N \curvearrowright \mathcal{H} \ni \xi_\tau$ CYCLIC FOR N.

BY UNIQUENESS OF GNS, $\mathcal{H} = L^2(N, \tau|_N)$

$L^2(N) \subset L^2(M)$. LET $e_N : L^2(M) \to L^2(N)$ PROJ

CLAIM: $\forall x \in M, \quad e_N(\hat{x}) \in \hat{N}$

LEM: LET (M, τ) VN ALG WITH TRACE, $\xi \in L^2(M)$. TFAE

(1) ξ IS LEFT BOUNDED: $\exists\, C > 0, \forall x \in M \ \|\xi . x\|_2 \leq C \|\hat{x}\|_2$

(2) $\xi \in \hat{M}$.

PF: (2) $\Rightarrow$ (1): LET $\xi = \hat{a}$, $a \in M$. THEN

$\|\xi . x\| = \| J x^* J a \xi_\tau \|_2 = \| a J x^* J \xi_\tau \|_2 \leq \underbrace{\|a\|_\infty}_{= C} \|\hat{x}\|_2$

OPERATOR NORM

(1) $\Rightarrow$ (2). SINCE ξ LEFT BOUNDED $L_\xi : \hat{M} \ni x \longmapsto \xi . x \in L^2(M)$ EXTENDS TO A BOUNDED MAP $L_\xi : L^2(M) \circlearrowleft$

CLAIM: $L_\xi \in M$. PF: LET $x, y \in M$. $L_\xi J x J \hat{y} = L_\xi \widehat{y x^*}$

$= J x y^* J \xi$

$J x J L_\xi \hat{y} = J x J J y^* J \xi$

$\Rightarrow L_\xi \in (JMJ)' = M'' = M$.

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THEN $\zeta = L_\zeta \hat{1} \in \hat{M}$.

$e_N(\hat{x})$ is LEFT-BOUNDED IN $L^2(N)$. $\widehat{ba^*} \in \hat{N}$

LET $a, b \in N$. $\langle Ja^*J e_N(\hat{x}), \hat{b}\rangle = \langle e_N(\hat{x}), JaJ\hat{b}\rangle$

$= \langle \hat{x}, e_N JaJ\hat{b}\rangle$

$= \langle Ja^*Jx\zeta_\tau, \hat{b}\rangle$

$\boxed{\forall x \in M,\ e_N(\hat{x}) \in \hat{N}}$

$= \langle xJa^*J\zeta_\tau, \hat{b}\rangle$

$= \langle x\hat{a}, \hat{b}\rangle$

$|\langle Ja^*Je_N(\hat{x}), \hat{b}\rangle| \le \|x\|_\infty \|\hat{a}\|_2 \|\hat{b}\|_2$

$\Rightarrow \|Ja^*Je_N(\hat{x})\|_2 \le \|x\|_\infty \|\hat{a}\|_2$.

$\Rightarrow \boxed{e_N(\hat{x}) = \widehat{E_N(x)}}$ FOR SOME UNIQUE $E_N : M \to N$

CHECK E_N LINEAR, POSITIVE.

J PRESERVES $L^2(N)$ $\quad e_N : L^2(M) \to L^2(N) = \overline{N\zeta_\tau}$

$\Rightarrow Je_N = e_N J$

$\Rightarrow Je_N(\hat{x}) = e_N(J\hat{x}) \iff \widehat{E_N(x)^*} = \widehat{E_N(x^*)}$

$E_N(x)^* = E_N(x^*)$

Claim: $a \in N$, $x \in M$, $E_N(ax) = a E_N(x)$

LET $y \in N$, $\langle \widehat{E_N(ax)}, \hat{y} \rangle = \langle e_N(\widehat{ax}), \hat{y} \rangle$

$$= \langle \widehat{ax}, \underbrace{e_N(\hat{y})}_{= \hat{y}} \rangle = \langle \hat{x}, \underbrace{\widehat{a^* y}}_{\in \hat{N}} \rangle$$

$$= \langle e_N(\hat{x}), \widehat{a^* y} \rangle$$

$$= \langle a E_N(\hat{x}), \hat{y} \rangle.$$

$E_N(ax) = a E_N(x)$. TAKE $*$: $a \to b^*$, $x \to x^*$

$$E_N(xb) = E_N(x) b.$$

(2) $\tau(E_N(x)) = \langle e_N(\hat{x}), \hat{1} \rangle = \langle \hat{x}, \underbrace{e_N(\hat{1})}_{= \hat{1}} \rangle = \tau(x)$.

$$R = \pi_\tau(A)'', \quad A = \overline{\bigcup_{n=1}^{\infty} M_2(\mathbb{C})^{\otimes n}}^{\|\cdot\|}$$

Def: M vN Alg is HYPERFINITE if $\exists M_1 \subset M_2 \subset \ldots$ f.dim $*$-SUBALG OF M s.t. $M = \overline{\bigcup_{n=1}^{\infty} M_n}^{\tau\varepsilon}$

Fact: (MURRAY-VN)
- HYPERFINITE II_1-FACTOR IS UNIQUE UP TO $*$-ISOMORPHISM (i.e. ALWAYS $\cong R$).
- SUBFACTOR OF R IS ALSO HYPERFINITE.