INTRODUCTION TO THE HYPERFINITE II_1 FACTOR AND STANDARD FORM

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REF: SUNDER: AN INVITATION TO VON NEUMANN ALG'S

LET H: HILBERT SPACE / $\mathbb{C}$, $\langle \, , \rangle$ INNER-PRODUCT (LINEAR, CONJ. LINEAR)

$\mathbb{B}(H) = \{ x : H \to H \mid$ BOUNDED (NORM-CONTINUOUS) LINEAR OPERATORS $\}$

$$\forall x \in \mathbb{B}(H) \quad \exists!\, x^* \text{ s.t.}$$

$$\forall \xi, \forall \eta \in H \quad \langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle$$

AND $(x+y)^* = x^* + y^*$

$$(x^*)^* = x$$

$\rightsquigarrow \mathbb{B}(H)$ is A $*$-ALGEBRA : $x+y$, xy, x^* WITH UNIT 1_H.

<u>OPERATOR TOPOLOGY</u>: NORM $\|x\| = \sup_{\substack{\|\xi\| \le 1 \\ \xi \in H}} \|x\xi\|$

$\rightsquigarrow$ BANACH SPACE

- SEQ $x_n, x \in \mathbb{B}(H)$

① $x_n \xrightarrow{\text{SOT}} x$ (STRONG OPERATOR TOPOLOGY)

$$\iff \forall \xi \in H, \quad \|x_n\xi - x\xi\| \longrightarrow 0$$

② $x_n \xrightarrow{WOT} x$ (WEAK OPERATOR TOPOLOGY)

$\Leftrightarrow \forall \xi, \eta \quad \langle x_n \xi, \eta \rangle \longrightarrow \langle x\xi, \eta \rangle$

NORM TOP > SOT > WOT

Def: LET $A \subset B(H)$ BE A $*$-SUBALG ($1_H \in$)

(1) A is A C^*-ALG IF $\overline{A}^{\|\cdot\|} = A$ (NORM CLOSED)

(2) A IS A VON NEUMANN ALG IF $\overline{A}^{SOT} = A$ (Π)

(3) FOR $S \subset \mathbb{B}(H)$, DENOTE $S' = \{x \in \mathbb{B}(H) \mid \forall s \in S, xa = ax\}$ = COMMUTANT OF S

THM (vN) FOR $*$-ALG $M \subset \mathbb{B}(H)$,

$$\overline{M}^{SOT} = M'' = (M')'$$

PROOF: ① $\overline{M} \subset M''$: LET $x \in \overline{M} \Rightarrow \exists x_i \in M,\ x_i \xrightarrow{SOT} x$

LET $y' \in M'$, $\xi \in H$. THEN $xy'\xi = \lim_i x_i y'\xi = \lim_i y' x_i \xi$
$= y' x \xi$

$\Rightarrow x \in M''$.

② $\overline{M} \supset M''$: LET $x \in M''$ ($x \in \overline{M} \Leftrightarrow \forall \varepsilon > 0, \forall \xi_1, \ldots, \forall \xi_n \in H$
$\exists x_0 \in M$ s.t. $\| x_0 \xi_i - x\xi_i \| < \varepsilon \quad i = 1, \ldots, n$)

LET $H_n = \underbrace{H \oplus \dots \oplus H}_{n}$ "AMPLIFIED HILBERT SPACE"

FOR $a \in \mathbb{B}(H)$, $\pi(a)(\eta_1, \dots, \eta_n) := (a\eta_1, \dots, a\eta_n)$

$$\begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \in \mathbb{B}(H_n)$$

LET $E = \overline{\{\pi(a)(\xi_1, \dots, \xi_n) \mid a \in M\}} \in H_n$

e = PROJ OF H_n ONTO E

- E is $\pi(M)$-INVARIANT
- $E^\perp$ is $\pi(M)$-INVARIANT

$\Rightarrow$ e COMMUTES WITH $\pi(y)$ $\forall y \in M$

$\Rightarrow e = [e_{ij}]$ $e_{ij} \in M'$.

SINCE $x \in M''$, $\pi(x) = \begin{bmatrix} x & & \\ & \ddots & \\ & & x \end{bmatrix}$ COMMUTES WITH e

$\Rightarrow \pi(x)\vec{\xi} = \pi(x) e \vec{\xi} = e\pi(x)\vec{\xi} \in E$

EXERCISE: WE ARE DONE !

DEF: • M vN-ALG $\subset \mathbb{B}(H)$. M is CALLED A FACTOR if $Z(M) = M \cap M' = \mathbb{C} 1_H$.

• M is A II_1-FACTOR if ① M is A FACTOR

② $\dim_{\mathbb{C}} M = \infty$

③ $\exists\, \tau : M \to \mathbb{C}$ FAITHFUL NORMAL TRACIAL ~~STATE~~.

RECALL: FOR $x = x^* \in \mathbb{B}(H)$, TFAE:

(1) $Sp(x) = \{\lambda \in \mathbb{C} \mid x - \lambda 1_H$ is NOT INVERTIBLE IN $\mathbb{B}(H)\}$ $\subset [0, \infty)$

(2) $\exists\, a \in \mathbb{B}(H)$ $x = a^* a$

(3) $\exists!\, b = b^* \in \mathbb{B}(H)$ s.t. $x = b^2$ AND $Sp(b) \subset [0, \infty)$
$\longrightarrow b = x^{1/2}$

(4) $\forall \xi \in H$, $\langle x\xi, \xi \rangle \geq 0$

DEF: LET $A \subset \mathbb{B}(H)$ BE A C^*-ALG. ($1 \in$)

$\varphi : A \xrightarrow{\text{LINEAR}} \mathbb{C}$ IS A STATE IF

(1) (POSITIVE) $\varphi(a) \geq 0$ $\forall a \in A_+ = \{x \in A \mid x \geq 0\}$

(2) $\varphi(1) = 1$.

REM: A STATE IS NORM-CONTINUOUS, $\|\varphi\| = \varphi(1)$.

A STATE IS TRACIAL IF $\varphi(ab) = \varphi(ba)$ $\forall a, b \in A$.

φ IS FAITHFUL IF $\varphi(a) = 0$, $a \geq 0 \implies a = 0$.

DEF: M vN ALG, $\varphi : M \to \mathbb{C}$ IS NORMAL IF IT IS WOT-CONTINUOUS ON $\mathrm{BALL}(M) = \{x \in M \mid \|x\| \leq 1\}$.

THM (GNS) LET A BE A C^*-ALG.
φ STATE ON A. THEN $\exists (H_\varphi, \xi_\varphi, \pi_\varphi)$ WITH

H_φ : HILBERT SPACE

$\xi_\varphi \in H_\varphi$, $\|\xi_\varphi\| = 1$

$\pi_\varphi : A \xrightarrow{*\text{-Hom}} \mathbb{B}(H_\varphi)$

s.t. ① $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi \rangle$

② $\{\pi_\varphi(a)\xi_\varphi ; a \in A\}$ is DENSE in H_φ.

(ξ_φ : CYCLIC FOR $\pi_\varphi(A)$) :

MOREOVER, THIS TRIPLE IS UNIQUE : IF $\exists (H, \xi, \pi)$ SATISFYING ①, ② THEN $\exists\, V : H_\varphi \xrightarrow{\sim} H$ UNITARY

s.t. $\begin{cases} V\pi_\varphi(a)V^* = \pi(a), & a \in A \\ V\xi_\varphi = \xi \end{cases}$

OUTLINE OF PROOF : REGARD A AS A (PRE-) INNER-PRODUCT SPACE BY $\langle a, b\rangle_\varphi := \varphi(b^* a)$

$N_\varphi = \{ a \in A \mid \|a\|_\varphi = \langle a, a\rangle_\varphi^{1/2} = 0 \}$

BY CAUCHY-SCHWARTZ, $|\varphi(b^*a)| \leq \varphi(b^*b)^{1/2}\varphi(a^*a)^{1/2}$

$\leadsto N_\varphi$ is A VECTOR SPACE.

$$H_\varphi = \overline{A/N_\varphi}^{\|\cdot\|_\varphi}$$

$$A \ni a \longmapsto \hat{a} \in H_\varphi$$

$$\hat{A} = \{\hat{a} \mid a \in A\} \underset{\text{DENSE}}{\subset} H_\varphi$$

$$\xi_\varphi = \hat{1} \in H_\varphi \qquad \|\xi_\varphi\|^2 = \langle 1, 1\rangle_\varphi = \varphi(1) = 1$$

DEFINE $\underset{a \in A}{\pi_\varphi^\circ(a)} : \hat{A} \longrightarrow \hat{A}$ BY $\pi_\varphi^\circ(a)\,\hat{b} = \widehat{ab}$

NOW $\|\widehat{ab}\|_\varphi^2 = \varphi(b^*a^*ab) \leq \|a\|^2 \|\hat{b}\|_\varphi^2$

HAVE $b^*a^*ab \leq \|a\|^2 b^*b$ BECAUSE

$$\langle b^*a^*ab\eta, \eta\rangle = \|ab\eta\|^2 \leq \|a\|^2\|b\eta\|^2$$
$$= \|a\|^2 \langle b^*b\eta, \eta\rangle$$

$\rightsquigarrow \pi_\varphi^\circ(a)$ EXTENDS TO $\pi_\varphi(a) : H_\varphi \longrightarrow H_\varphi$

$\|\pi_\varphi(a)\| \leq \|a\|$.

$$\langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle = \langle \hat{a}, \hat{1}\rangle_\varphi = \varphi(a)$$

$$\pi_\varphi(A)\xi_\varphi = \{\hat{a},\ a \in A\} \underset{\text{DENSE}}{\subset} H_\varphi.$$

IF (H, π, ξ) SATISFY ①, ②

$$V : \underset{a \in A}{\pi_\varphi(a)\xi_\varphi} \longmapsto \pi(a)\xi$$

$$\left\| \sum_{i=1}^{N} \pi_\varphi(a_i)\,\xi_\varphi \right\|^2 = \sum_{i,j=1}^{N} \langle \pi_\varphi(a_i)\xi_\varphi, \pi_\varphi(a_j)\xi_\varphi \rangle$$

$$= \sum_{i,j} \langle \pi_\varphi(a_j^* a_i)\xi_\varphi, \xi_\varphi \rangle$$

$$= \sum_{i,j} \varphi(a_j^* a_i) = \sum_{i,j} \langle \pi(a_j^* a_i)\xi, \xi \rangle$$

$$= \dots = \left\| \sum_{i,j}^{N} \pi(a_i)\xi \right\|$$

$\rightsquigarrow$ V EXTENDS TO AN ISOMETRY $H_\varphi \longrightarrow H$

V is ONTO. $V\xi_\varphi = \xi$.

CONSTRUCTION OF HYPERFINITE II_1 FACTOR

$$M_2(\mathbb{C}) \hookrightarrow M_2(\mathbb{C})^{\otimes 2} \hookrightarrow M_2(\mathbb{C})^{\otimes 3} \hookrightarrow \dots$$

$$M_2^{\otimes n} \ni x \longmapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in M_2^{\otimes(n+1)} \cong M_2\left(M_2^{\otimes n}\right) \cong M_{2^{n+1}}(\mathbb{C})$$

$$A = \overline{\bigcup_{n=1}^{\infty} M_2(\mathbb{C})^{\otimes n}}^{\|\cdot\|} \qquad C^*\text{-ALG}$$

$$\forall n,\ \exists\, \tau_n : M_2^{\otimes n} \longrightarrow \mathbb{C}$$
$$\tau_n = \frac{1}{2^n}\mathrm{Tr}(\cdot)$$

$\tau_{n+1}\big|_{M_2^{\otimes n}} = \tau_n \qquad \exists!\ \tau$ TRACE ON A, $\tau\big|_{M_2^{\otimes n}} = \tau_n$

GNS (H, π, ξ), $\tau(a) = \langle \pi(a)\xi, \xi \rangle$.

DEF: $R = \pi(A)''$ HYPERFINITE II_1 FACTOR.