WHAT IS A QUANTUM GROUP?

QUANTUM GROUP $\subseteq$ NCG

$\mathcal{C}$ CATEGORY OF "SPACES"

$\mathcal{F}(\mathcal{C})$ ALG OF FUNCTIONS ON $X \in \mathcal{C}$

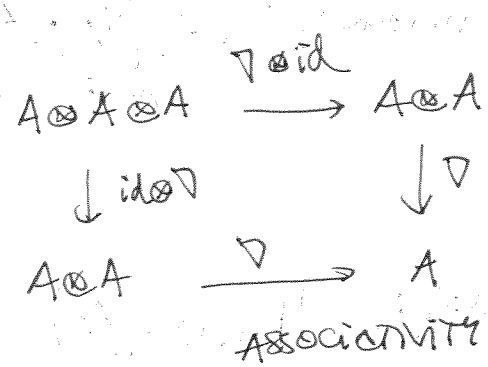
G GROUP $\rightsquigarrow$ $\mathcal{F}(G)$

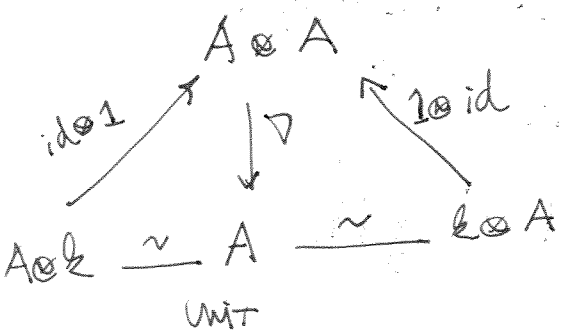
QUANTUM GROUP $\longrightarrow$ HOPF ALGEBRAS

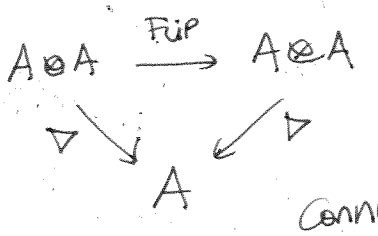
= "OPPOSITE CATEGORY OF THE CATEGORY OF HOPF ALGEBRAS"

ALG $(A, \nabla, 1)$

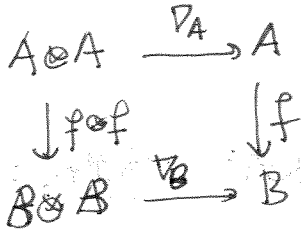
MULTIPLICATION $A \otimes A \to A$; UNIT $k \to A$







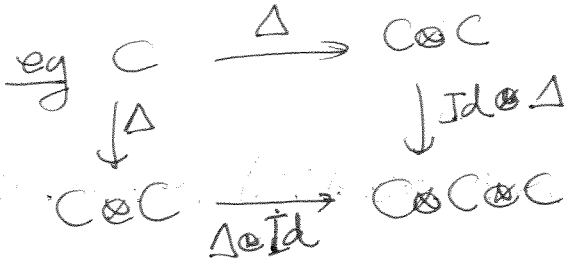
$(A, \nabla_A, 1_A) \xrightarrow{f} (B, \nabla_B, 1_B)$ SATISFIED



COALGEBRAS: $(C, \Delta: C \to C \otimes C, \varepsilon: C \to k)$

COPRODUCT. COUNIT

SATISFIES DUAL PROPERTIES.



EXAMPLES

(1) A ALGEBRA (FINITE-DIM) $\Rightarrow A^*$ COALGEBRA

$\nabla: A\otimes A \to A \rightsquigarrow A^* \xrightarrow{\nabla^*} (A\otimes A)^* \cong A^*\otimes A^*$

$1: k \to A \rightsquigarrow 1^*: A^* \to k^* \cong k$

(2) k, $\varepsilon: k \to k$, $1 \mapsto 1$, $\Delta(1) = 1\otimes 1$

(3) $(C, \Delta_C, \varepsilon_C)$, $(D, \Delta_D, \varepsilon_D)$ COALGEBRAS $\to C\otimes D$ COALG WITH

$\Delta = (id\otimes Flip\otimes id)\circ(\Delta_C\otimes\Delta_D): C\otimes D \to C\otimes C\otimes D\otimes D \to C\otimes D\otimes C\otimes D$

BIALGEBRAS: $(B, \nabla, 1, \Delta, \varepsilon)$ (ALG, COALG) s.t. (1) $\nabla, 1$ ARE COALG-MORPHISMS

(2) Δ, ε ARE ALG-MORPHISMS

(ACTUALLY, (1) $\Leftrightarrow$ (2))

REM: $\mathrm{Hom}(C, A)$ (COALG, ALG), $*$, 1ε — $C \xrightarrow{\varepsilon} k \xrightarrow{1} A$, 1ε

$f, g \in \mathrm{Hom}(C, A)$,

$C \xrightarrow{\Delta} C\otimes C \xrightarrow{f\otimes g} A\otimes A \xrightarrow{\nabla} A$

$f * g$

HOPF ALGEBRA: $(H, \nabla, 1, \Delta, \varepsilon, S)$, $S \in \mathrm{Hom}(H, H)$, $S * id_H = id_H * S = 1\varepsilon$

ANTIPODE

EX: (1) G FOR GROUP, $\mathcal{F}(G)$ $\Delta: \mathcal{F}(G) \to \mathcal{F}(G)\otimes\mathcal{F}(G) \cong \mathcal{F}(G\times G)$

$f \mapsto [\Delta f: x\otimes y \mapsto f(xy)]$

(2) (MORE SPECIFIC) $G = SL_2\mathbb{C}$ ALGEBRAIC GROUP

→ CHARACTERIZED BY ITS COORDINATE RING

$\mathcal{F}(G) = \mathbb{C}[X_{ij}]/(\det - 1)$ $\quad (G = SL_2\mathbb{C})$

↑ POLYNOMIALS IN 4 VARIABLES (COMMUTING)

DEFORMATION: $\mathcal{F}_h(G) = \mathbb{C}\{a,b,c,d\}[\![h]\!]/\text{RELATIONS}$

↑ POLYNOMIALS ON 4 NON-COMM. VARIABLES

RELATIONS: $ab = e^h ba$, $ac = e^{-h} ca$, $bd = e^{-h} db$, $cd = e^h dc$

$bc = cb$, $ad - da = (e^{-h} - e^h)\, bc$, $ad - e^{-h} bc = 1$

$\mathcal{F}_h(SL_2\mathbb{C})$ IS A HOPF ALGEBRA:

$$\Delta_h\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$\Delta(a) = a \otimes a + b \otimes c$ $\quad$ (NOT COCOMMUTATIVE)

$$S_h\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -e^h b \\ -e^{-h} c & a \end{pmatrix}$$

$$\varepsilon_h\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

REM: $\Delta : \mathcal{F}_h(SL_2\mathbb{C}) \to \mathcal{F}_h \hat{\otimes} \mathcal{F}_h$ (↑ COMPLETED)

(3) $(\mathfrak{g}, [,])$ LIE ALG, $U(\mathfrak{g})$ UNIVERSAL ENVELOPING ALG

$\mathfrak{g} \xrightarrow{\text{Lie}} U(\mathfrak{g}) \downarrow$, $\mathfrak{g} \xrightarrow{\text{Lie}} A$

$\Delta X = X \otimes 1 + 1 \otimes X$

(SATISFIES $\Delta[X,Y] = [\Delta X, \Delta Y]$)

$\mathfrak{g} \xrightarrow{\Delta} U(\mathfrak{g}) \otimes U(\mathfrak{g})$

$\rightsquigarrow \exists \quad \uparrow \Delta$

$U(\mathfrak{g})$

$SX = -X$

$\varepsilon(X) = 0$

$\varepsilon(1) = 1$

(4) Cohomology of $\underline{H}$ Groups
↑ "Hopf"

- H-space: (X, e) with $X \times X \xrightarrow{\mu} X$ s.t. $(x \mapsto x \cdot e = \mu(x,e))$ and $(x \mapsto e\,x) \simeq id_X$
- Homotopy associative: $(x,y,z) \mapsto (xy)z \simeq (x,y,z) \mapsto x(yz)$
- Homotopy inverse: $\exists\, X \xrightarrow{I} X$ s.t. $[x \mapsto x\,I(x)]$ and $[x \mapsto I(x)\,x] \simeq [x \mapsto e]$

$\rightsquigarrow$ $H^*(X, k)$ is a Hopf Alg. (X path connected)

$\Delta = \mu^*$

$S = I^*$