CHRIS 8

EXERCISE: 1) (A, R, μ) CHARMED HOPF
$\Longrightarrow (A\text{-Mod}, \beta_R, \theta_V)$ RIBBON CATEGORY

2) $(\varepsilon \otimes 1)R = (1 \otimes \varepsilon)R = 1$.

THE CHARMED ELEMENT FOR $U_q^{res}\mathfrak{g}$.

IDEA: LOOKING FOR INVERTIBLE $\mu \in U_q^{res}\mathfrak{g}$.
POSTULATE $\mu = \prod K_i^{m_i}$. USE $S^2(a) = \mu a \mu^{-1}$ TO CONSTRAIN μ.

CALCULATION: $S^2(E_j) = \mu E_j \mu^{-1}$ $\quad S(E_j) = -E_j K_j^{-1}$

$$K_j E_j K_j^{-1} = \left(\prod K_i^{m_i}\right) E_j \left(\prod K_i^{-m_i}\right)$$

USING $S(E_j) = -E_j K_j^{-1}$, $K_i E_j K_i^{-1} = q^{d_i a_{ij}} E_j$

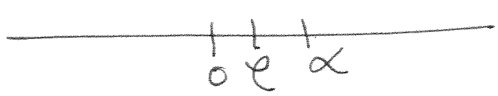
$$\rightsquigarrow S^2(E_j) = q^{d_j a_{ij}} \quad \prod q^{m_i d_i a_{ij}} = \mu E_j \mu^{-1}$$

SHOULD BE EQUAL

RECALL: $a_{ij} := \dfrac{\langle \alpha_i, \alpha_j \rangle}{d_i}$ $\to$ NEED $\langle \alpha_j, \alpha_i \rangle = \langle \sum m_i \alpha_i, \alpha_j \rangle \ \forall j$

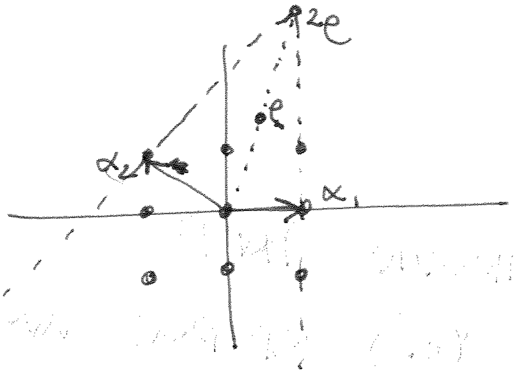
Def: $\rho \in \Lambda$ is THE UNIQUE WEIGHT s.t. $\langle \alpha_j, \alpha_i \rangle = \langle 2\rho, \alpha_j \rangle \ \forall j$

Eg: $\mathfrak{sl}_2$



$\mu = K$

SO(5)



$\rightsquigarrow \quad \mu = K_1^4 K_2^3$

CONCLUSION: THE CHARMED ELEMENT $\mu = \prod K_i^{m_i}$ WHERE $2\rho = \sum m_i \alpha_i$.

Eg RIBBON STRUCTURE FOR sl_2:

RECALL: RIBBON ELEMENT IS $\nu := m(1\otimes\mu)\tau(R)$ (CENTRAL, INVERTIBLE)

CENTRAL $\Rightarrow$ ACTS BY A SCALAR ON ANY IRRED REP W_λ

DETERMINE THAT SCALAR BY THE ACTION ON A HIGHEST WEIGHT VECTOR v_λ:

$$[m(1\otimes\mu)\tau(R)]v_\lambda = [m(1\otimes \underbrace{K}_{\mu=K})\Big(q^{\frac{1}{2}H\otimes H}\sum_{t\geq 0} q^{\binom{t}{2}}\frac{1}{[t]!}F^t\otimes E^t\Big)$$

(v_λ HIGHEST WEIGHT, E^t RAISING $\rightarrow$ ONLY $t=0$ TERM LEFT).

$$= \left[m(1\otimes K)\left(q^{\frac{1}{2}H\otimes H}\right)\right]v_\lambda = q^{\frac{1}{2}\lambda^2+\lambda}$$

(RECALL: $q^{\frac{1}{2}(H\otimes H)}: \Lambda\times\Lambda \to \mathbb{Z}[q^{\pm\frac{1}{2}}]$
$(\lambda,\mu)\mapsto q^{\frac{1}{2}H(\lambda)H(\mu)}$)

θ_i IN ANDRE'S HANDOUT.

THE FUSION CATEGORY

PLAN: $\underbrace{Rep^{tilt}}_{}$ CATEGORY OF $\underbrace{U_q^{res}\mathfrak{g}}$ CATEGORY OF TILTING MODULES

$W_{\lambda<l-1}$ SIMPLE $\in$

$W_{\lambda=l-1}$ SIMPLE

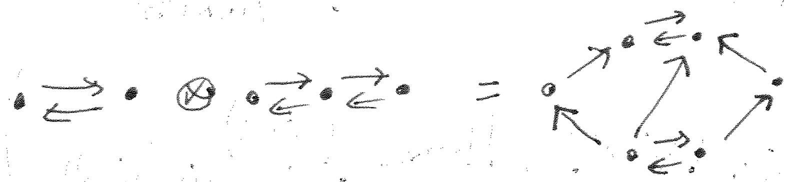
$\downarrow$ 0

W_λ (SIMPLE) $\in Rep^{tilt}\, U_q^{res}\mathfrak{g} /\sim$ WHICH IS SEMI-SIMPLE

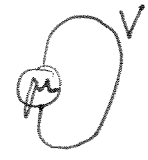
$W_{\lambda>l-1}$ REDUCIBLE IN Rep^{tilt}

$W_{\lambda<l-1} \otimes W_{\lambda<l-1} \cong \bigoplus W_{\lambda<l-1} \oplus \underbrace{\bigoplus N_j}_{\text{NOT SEMI-SIMPLE}}$

IDEA: MAP ALL THESE N_j'S TO 0.

eg: $q^6 = 1$  T_3

IDEA: KILL ALL INDECOMPOSABLE REDUCIBLE TILTING MODULES (ALL qdim 0)

QUANTUM DIMENSION: qdim: 

HERE $\mu = K$ $\mapsto$

q^{-3} q^{-1} q q^3

q^{-1} q

$\downarrow$ SUM

$[4]+[2]$

$= -1 + 1 = 0$.

$q^6=1$

NEGLIGIBLE MORPHISMS & THE SEMI-SIMPLE QUOTIENT

WANT TO MOD OUT BY SOME IDEAL OF MORPHISMS, IN FACT A "TENSOR IDEAL" TO MAKE SURE THE QUOTIENT IS STILL A TENSOR CATEGORY. THE IDEAL SHOULD INCLUDE $id_{T_{\lambda>l-1}}$.

Def: A morphism $f: V \to W$ is called <u>negligible</u> if $\underbrace{qtr}_{\text{quantum trace}}(fg) = 0 \quad \forall g: W \to V$.

Need: $id_{T_{\lambda > l-1}}$ is negligible

Eg: $q^6 = 1$. Want that $\forall f \in End(T_3) = \langle id, n \rangle$, $qtr(f) = 0$.

Already checked for id.

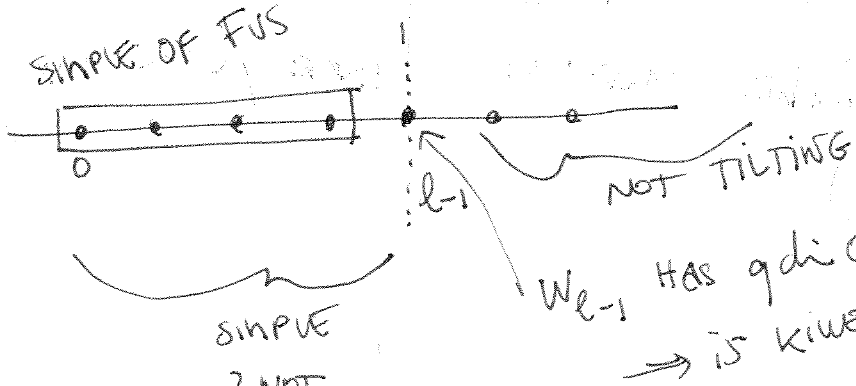
For n, $qtr(n) = tr(\mu^{-1} n) = 0$.

μ^{-1}: diagonal; n: strictly upper triangular

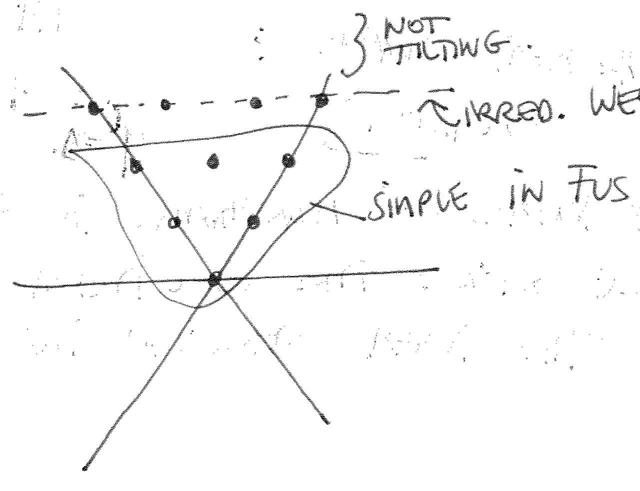
Def: $Fus\, U_q^{res}\mathfrak{g} = Rep^{tilt}\, U_q^{res}\mathfrak{g} / \text{negligible morph}$ = the fusion category of $U_q^{res}\mathfrak{g}$.

$$\left(Hom_{Fus}(V,W) = Hom_{Tilt}(V,W) / Neg(V,W) \right)$$

Eg: $\mathfrak{sl}_2$ ($q^{2l} = 1$, $l = 5$)



Eg $\mathfrak{sl}_3$:



NEED TO CHECK THAT THE CATEGORY SATISFIES THE MODULARITY CONDITION.

<u>THM</u>: $\mathrm{Fus}\, U_q^{\mathrm{res}}\mathfrak{g}$ AT q^{2l} is MODULAR IF $D | l$ AND

$$\frac{l}{D} \geq \langle \alpha_0, \rho \rangle + 1$$

$\uparrow$ HIGHEST ROOT

$D = \begin{cases} 1 & \text{TYPE ADE} \\ 2 & \text{TYPE BCF} \\ 3 & \text{TYPE G} \end{cases}$

eg: $\mathfrak{sl}_2$, $l \geq 2$.

Eg $\mathfrak{sl}_2$: EXERCISE 1^*: PROVE THAT $S_{nm} = \Big[$  $\Big]_{nm} = [(n+1)(m+1)]_q$

$n, m \in \{0, \dots, l-2\}$

EX 2: ~~COMPUTE~~ CONCLUDE THAT $\mathrm{Fus}\, U^{\mathrm{res}}\mathfrak{sl}_2$ is MODULAR.