CHRIS 7

THE RIBBON STRUCTURE ON THE QUANTUM GROUP

BALANCED HOPF ALGEBRAS AND BALANCED TENSOR CAT

QUESTION: WHAT STRUCTURE ON (A,R) MAKES $(A\text{-Mod},\beta)$ BALANCED?

APPROACH:
$$\begin{tikzcd} A\text{-Mod} \arrow[r, bend left, "id"] \arrow[r, bend right, "id"'] \arrow[r, phantom, "\Downarrow \text{“TWIST”}"] & A\text{-Mod} \end{tikzcd}$$

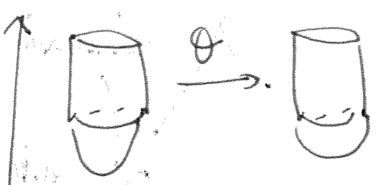
DEHN-TWIST ON THE CYLINDER
$\Leftrightarrow$ NAT TRANSFORMATION $id \to id$

$\forall M$, NEED $\begin{matrix} M \\ \downarrow \\ M \end{matrix}$. TRY $\begin{matrix} m \\ \downarrow \\ \nu m \end{matrix}$ FOR SOME $\nu \in A$ INVERTIBLE.

CONDITION: 1) A-MODULE MAP:
$$\begin{tikzcd} m \arrow[r, mapsto] \arrow[d, mapsto] & \nu m \arrow[d, mapsto] \\ am \arrow[r, mapsto] & \nu a m \arrow[u, phantom, "\|"] \end{tikzcd}$$
$a\nu m$

$\Rightarrow$ ν NEEDS TO BE CENTRAL

2) COMPATIBILITY OF UNIT AND TWIST:



$\rightsquigarrow$ $\mathbb{C} \xrightarrow{\theta_\nu} \mathbb{C}$

$m \longmapsto \nu . m = \varepsilon(\nu) m$

$\theta_1 = id_1$ (DEHN TWIST TRIVIAL WITH THE CAP)

(ACTION OF A ON $\mathbb{C}$ VIA ε)

$\Rightarrow$ NEED $\varepsilon(\nu) = 1$ (1)

3) COMPATIBILITY OF BRAIDING AND TWIST

$\beta^2 = \theta^{-1}(\theta \otimes \theta)$

$\left[\begin{matrix} M\otimes N \xrightarrow{\beta} N \otimes M \xrightarrow{\beta} M \otimes N \\ m \otimes n \longmapsto R(n\otimes m) \longmapsto R\tau(R)(m\otimes n) \end{matrix} \right.$

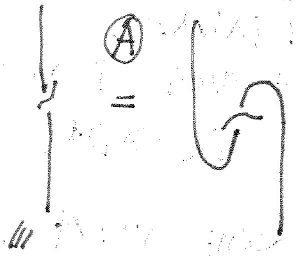
$= m\otimes n \longmapsto \Delta(\nu^{-1})(\nu m \otimes \nu n)$

⇝ NEED $\underline{R\tau(R) = \Delta(\nu^{-1})(\nu\otimes\nu)}$ (2)

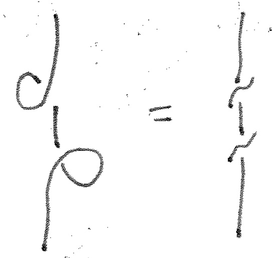
__Def__: A braided Hopf algebra (A,R) is __balanced__ given an invertible central $\nu \in A$ s.t. (1), (2).

__Ribbon Hopf algebras and ribbon tensor categories__

__Recall__: A balanced tensor cat is ribbon if



AND



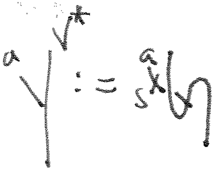
(illustrated with an actual belt...)

Twist θ

__Calculation__: Given a braided Hopf alg (A,R), and $V \in A\text{-mod}$, then

is given by

(__Recall__: ${}^{a}V^{*} :=$



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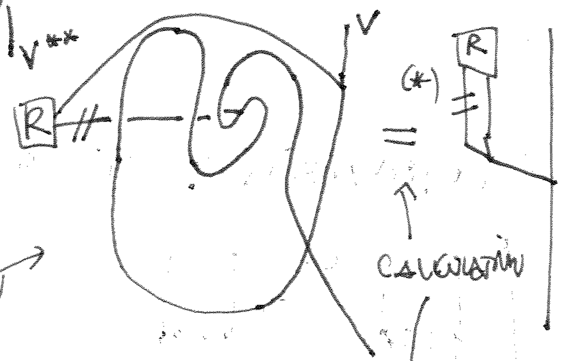


BY DEFINITION

CALCULATION

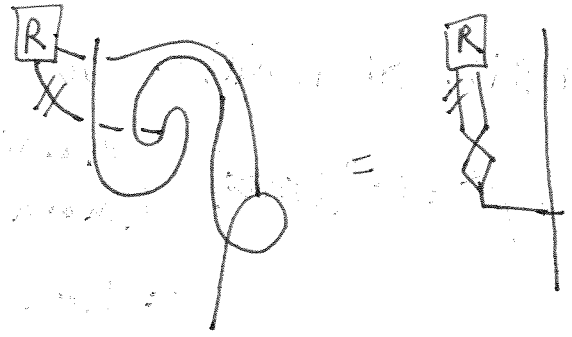
(CANCELLING DOUBLE DUALS)

__Def__: Given a braided Hopf alg, the __distinguished element__ is $u := m(S^2\otimes 1)R$

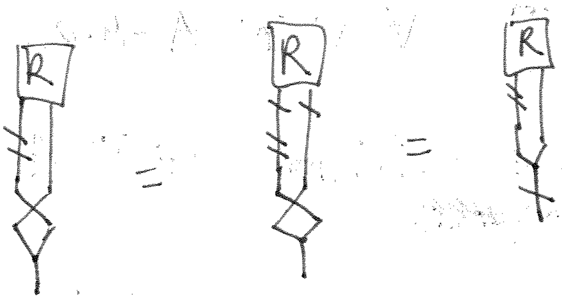
(element occuring in (*))

__Calculation__:

is given by



EXERCISE*: $(S \otimes S)R = R$

$\rightsquigarrow$  $= S(u)$

DISTINGUISHED ELEMENT

Def: A BALANCED HOPF ALG (A, R, ν) is RIBBON if

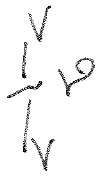
③ $S(\nu) = \nu$ $\quad (\Rightarrow ④)$

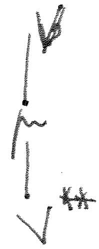
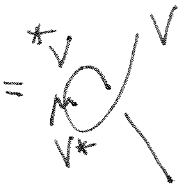
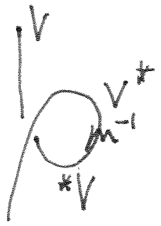
④ $S(u)u = \nu^2$ $\quad (\Rightarrow Ⓑ)$

CONCLUSION: FOR A RIBBON HOPF ALG (A, R, ν), A-Mod is A RIBBON TENSOR CATEGORY.

NOT EASY TO GIVE SUCH A ν FOR OUR QUANTUM GROUPS...

CHARMED HOPF ALG

IDEA: INSTEAD OF DIRECTLY DEFINING , WE'LL SUPPOSE

WE HAVE  AND DEFINE $\nu =$  $=$  $(**)$

μ is CALLED THE "CHARMED ELEMENT"

Def: GIVEN A BRAIDED HOPF ALGEBRA (A, R), AN INVERTIBLE ELEMENT $\mu \in A$ is CALLED CHARMED if

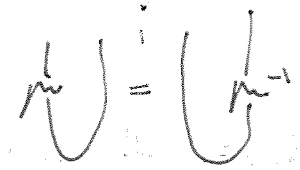
1) $\Delta(\mu) = \mu \otimes \mu$ $\qquad$ 3) $S^2(a) = \mu a \mu^{-1}$

2) $S(\mu) = \mu^{-1}$ $\qquad$ 4) $m(1 \otimes \mu^{-1})R = m(1 \otimes \mu)\tau(R)$ $(:= \nu)$

PRODUCT $A \otimes A \to A$

COMMENTS: 3) $\equiv$ LEFT MULTIPLICATION BY $\mu \in A$ IS AN A-Mod MAP $V \longrightarrow V^{**}$ $\forall$ V IN A-Mod

4) $\equiv$ THE TWO ABOVE DEFINITIONS OF ν AGREE (**)

NOTE: 3) $S^2(\mu) = \mu \xRightarrow{2)} S(\mu^{-1}) = \mu \xRightarrow{\text{(EXERCISE)}}$ 

PROP: (A, R, μ) CHARMED, THEN $\cancel{(A, R, \nu)}$ $(A\text{-Mod}, \beta, \theta_\nu)$ IS RIBBON.

WISH WE COULD JUST DO THAT BUT WILL INSTEAD

FACT: A RIBBON HOPF ALGEBRA $\Longrightarrow$ ASSOCIATED CHARMED HOPF ALGEBRA

? (PROBABLY, AT LEAST IF A f.d.)