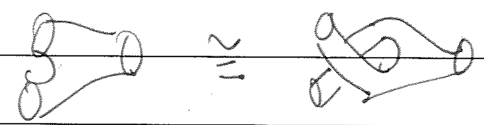
KØBENHAVNS UNIVERSITET
UNIVERSITY OF COPENHAGEN

WWW.KU.DK

CHRIS 5

THE QUANTUM GROUP AS A BRAIDED HOPF ALGEBRA

BRAIDED HOPF ALGEBRAS AND BRAIDED TENSOR CAT



QUESTION: WHAT STRUCTURE ON A HOPF ALG A ENSURES THAT A-Mod IS BRAIDED?

NEED:

$$\begin{tikzcd} A\text{-Mod} \otimes A\text{-Mod} \arrow[rr, "c"] \arrow[dr] & \arrow[d, phantom, "\Downarrow\beta"] & A\text{-Mod} \otimes A\text{-Mod} \arrow[dl] \\ & A\text{-Mod} & \end{tikzcd}$$

ie.

$$\begin{tikzcd} M \boxtimes N \arrow[r, mapsto] \arrow[d] & N \boxtimes M \arrow[d] \\ M \otimes N \arrow[r, dashed] & N \otimes M \end{tikzcd}$$

ATTEMPT 1: $m \otimes n \mapsto n \otimes m$ A-Mod map?

CHECK: $a(m\otimes n) = \Delta(a)(m \otimes n) \mapsto \tau(\Delta(a)(n \otimes m))$
$\overset{?}{=} \Delta(a)(n \otimes m)$

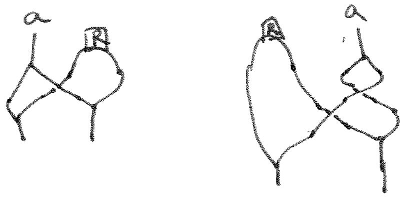
YES, IF Δ IS COCOMMUTATIVE, BUT NO IN GENERAL!

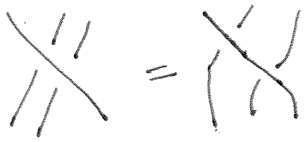
<u>ATTEMPT 2</u>: $m\otimes n \longmapsto R(n\otimes m)$ FOR SOME $R \in A\otimes A$ INVERTIBLE.

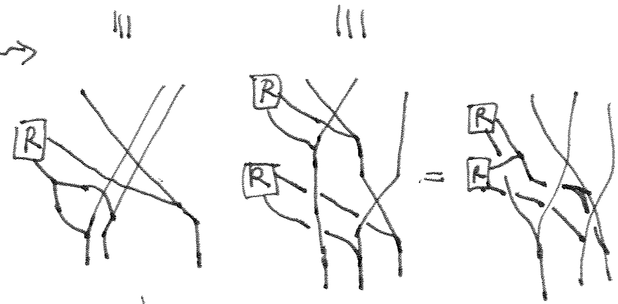
R = "UNIVERSAL R-MATRIX"

NOW $\Delta(a)(m\otimes n) \longmapsto R\tau(\Delta(a))(n\otimes m) \overset{?}{=} \Delta(a)\,R(n\otimes m)$

<u>CONDITIONS</u>: ① $\Delta(a)R = R\tau(\Delta(a))$ (ENSURES β WELL-DEFINED)



FOR β TO BE A BRAIDING, NEED 

WHERE $\diagup\!\!\!\!\diagdown := \begin{matrix} m\otimes n \\ \downarrow \\ R(n\otimes m)\end{matrix} \equiv$  $\rightsquigarrow$ 

$\rightsquigarrow$ ② $(\Delta\otimes 1)R = (1\otimes R)\left(\overset{R}{\otimes 1 \otimes}\right)$

ALSO $(1\otimes\Delta)R = (R\otimes 1)\left(\overset{R}{\otimes 1 \otimes}\right)$

<u>Def</u>: A HOPF ALG IS BRAIDED (AKA QUASI-TRIANGULAR) GIVEN $R \in A\otimes A$ INVERT. s.t. ① AND ②

<u>THE BRAIDED STRUCTURE ON $U_q\mathfrak{g}$</u>

DEUS EX MACHINA: FOR $U_q\mathfrak{sl}_2$: $R = q^{\frac{1}{2}H\otimes H}\sum_{t\geq 0} q^{\binom{t}{2}}\frac{1}{[t]_q!}E^t\otimes F^t$
(IF ONE CAN MAKE SENSE OF THE FORMULA)

<u>IDEA</u>: $\exists$ <u>HOPF PAIRING</u> BETWEEN $U_q\mathfrak{b}_+$ AND $(U_q\mathfrak{b}_-)^{op}$ ($\mathfrak{b}_+, \mathfrak{b}_-$ THROW AWAY THE E's OR F's)

($A\otimes B \longrightarrow \mathbb{C}$ s.t. $\langle a, b_1 b_2\rangle = \langle \Delta a, b_1\otimes b_2\rangle$)

FORMAL

$R = \sum e_p\otimes e^p$ WHERE $\{e_p\}$ IS A BASIS FOR $U_q\mathfrak{b}_+$.

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ISSUES!: 1) THE SUM IS INFINITE, SO NOT IN $U \otimes U$.

SOLUTION 1: REDEFINE TO ALLOW $R \in A \hat{\otimes} A$

SOLUTION 2: RESTRICT ATTENTION TO FINITE DIM REPS
($\rightsquigarrow$ WILL ONLY ACT NON-TRIVIALLY WITH FINITELY MANY TERMS)

2) R INVOLVES DIVIDING BY $[t]_q!$ WHICH IS FINE IF q GENERIC, BUT NOT IF $q^{2l}=1$...

3) R INVOLVES FRACTIONAL POWERS OF q, NOT IN OUR BASE RING

4) $q^{\frac{1}{2} H \otimes H}$ IS NOT ALGEBRAIC IN $K^{1/2}$

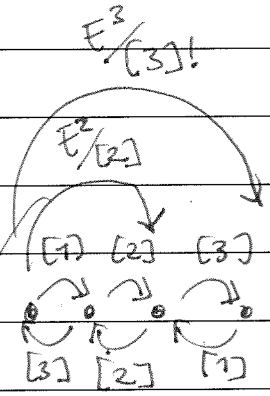
FUNCTION $\Lambda \to \mathbb{Z}[q^{1/2}, q^{-1/2}]$

FUNCTION ON $\Lambda \times \Lambda \to \mathbb{Z}[q^{\pm 1/2}]$

$(\lambda, \mu) \mapsto q^{\frac{1}{2} H(\lambda) H(\mu)}$

THE RESTRICTED QUANTUM GROUP

RESOLVE (2): OVER $\mathbb{C}(q)$, REPS LOOK LIKE



OPERATORS TAKE THE FIRST ACT TO THE SECOND / THIRD.
LIVE IN $U_q\mathfrak{sl}_2/\mathbb{C}(q)$

$U_q\mathfrak{g} \rightsquigarrow U_q^{res}\mathfrak{g} := \left\langle \frac{E_i^k}{[k]_{q_i}!}, \frac{F_i^k}{[k]_{q_i}!}, K^{\pm 1} \right\rangle$ $/\mathbb{Z}[q,q^{-1}]$

$\mathbb{Z}[q,q^{-1}]$ MODULE $\subset U_q\mathfrak{g}/\mathbb{C}(q)$

$\rightsquigarrow$ ADD THESE OPERATORS

WRITE $E_i^{(k)} := \frac{E_i^k}{[k]_{q_i}!}$ AND THINK OF IT AS A FORMAL OBJECT WHEN SPECIALIZING q

$$\rightsquigarrow R = q^{\frac{1}{2} H\otimes H} \sum_{t\geq 0} q^{\binom{t}{2}} [t]_q!\, E^{(t)} \otimes F^{(t)} \in \text{“} U_q^{res} \hat{\otimes} U_q^{res} \text{”}$$

<u>RESOLVE 3</u>: $U_q^{res}\mathfrak{g} \rightsquigarrow U_q^{res'}\mathfrak{g} := U_q^{res}\mathfrak{g} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}[q^{\frac{1}{L}}, q^{-\frac{1}{L}}]$

WHERE $L = \left| \dfrac{\text{WEIGHT LATTICE}}{\text{ROOT LATTICE}} \right|$

<u>RESOLVE 4</u>: $U_q'\mathfrak{h} \rightsquigarrow U_q^{+'}\mathfrak{h} := \mathrm{Map}(\Lambda, \mathbb{Z}[q^{\pm\frac{1}{L}}])$

<u>NOTE</u>: $q^{\frac{1}{2} H\otimes H} \in U_q^{+'}\mathfrak{h} \hat{\otimes} U_q^{+'}\mathfrak{h}$

THEN $U_q^{res'}\mathfrak{g} \rightsquigarrow U_q^{res+'}\mathfrak{g} = \text{“}\left(U_q^{res'}\mathfrak{g}\right)^{\wedge}\ U_q'\mathfrak{h} \rightsquigarrow U_q^{+'}\mathfrak{h}\text{”}$

<u>TABLE OF QUANTUM GROUP VARIATIONS</u>

$$\begin{tikzcd}
\text{SPECIALIZED} & \text{INTEGRAL} \arrow[l, squiggly] \arrow[r, squiggly] & \text{GENERIC} & \\
U_q\mathfrak{g}\ (q^\ell = 1) \arrow[d, "+\text{ DIVIDED POWERS}"'] & U_q\mathfrak{g}/\mathbb{Z}[q,q^{-1}] \arrow[l] \arrow[r] & U_q\mathfrak{g}/\mathbb{Q}(q) \text{ OR } \mathbb{C}(q) \arrow[d, equal] & \text{NOT BRAIDED} \\
\boxed{U_q^{res}\mathfrak{g}\ (q^\ell = 1)} \arrow[d, equal, "+q^{1/L}"'] & U_q^{res}\mathfrak{g} \arrow[u, hook] \arrow[l] \arrow[r] & U_q\mathfrak{g}/\mathbb{Q}(q) & \text{ALMOST BRAIDED} \\
U_q^{res'}\mathfrak{g}\ ((q^{\frac{1}{L}})^{\ell L} = 1) \arrow[d, "\text{COMPLETE}"'] & U_q^{res'}\mathfrak{g} \arrow[u, hook] \arrow[l] \arrow[r] \arrow[d] & U_q'\mathfrak{g}/\mathbb{Q}(q) \arrow[u, hook] & \text{MORALLY BRAIDED} \\
U_q^{res+'}\mathfrak{g} & U_q^{res+'}\mathfrak{g} \arrow[l] & & \text{ACTUALLY BRAIDED}
\end{tikzcd}$$

ESSENTIALLY GOOD ENOUGH