CHRIS 4

QUESTION: WHAT STRUCTURE ON $A \rightsquigarrow$ A-Mod HAS DUALS?

APPROACH: WANT $A \circlearrowright V^* := \mathrm{Hom}(V, \mathbb{C})$ FOR V IN A-Mod

DEF: $(af)(v) := f(s(a)v)$ FOR SOME $s : A \to A$

CONDITIONS NEEDED ON s:

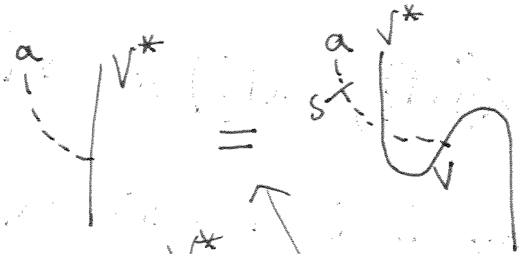
- NEED $(ab)(f)(v) = a(bf)(v)$

$$\begin{array}{ccc} (ab)(f)(v) & = & a(bf)(v) \\ \| & & \| \\ f(s(ab)v) & & f(s(b)s(a)v) \end{array}$$

$\Rightarrow$ s IS AN ALGEBRA ANTI-HOMOMORPHISM

- NEED $ev : V^* \otimes V \longrightarrow \mathbb{C}$, $coev : \mathbb{C} \longrightarrow V \otimes V^*$ TO BE MODULE MAPS.

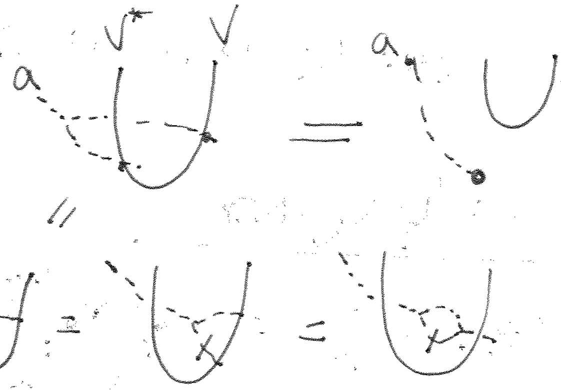
NOTATION:



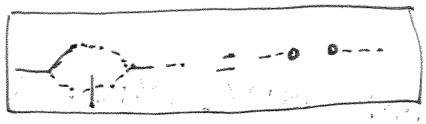
a ACTS ON V^*

HOW WE DEFINED THE ACTION

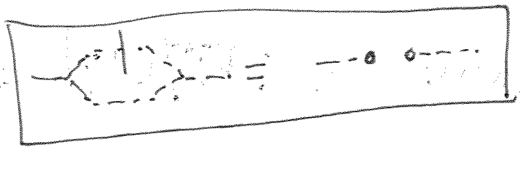
ev MODULE MAP: NEED



$\rightsquigarrow$ HOPF CONDITION 1:



COEV MODULE MAP $\rightsquigarrow$ HOPF COND 2:



Def: A Hopf algebra is a bialgebra with an invertible algebra / coalg anti-homomorphism $S : A \to A$ satisfying Hopf conditions 1 and 2

Conclusion: A Hopf $\Rightarrow$ A-Mod is a tensor category with duals.

The Hopf structure on $U_q\mathfrak{g} = A$:

Need
$$\Delta : A \longrightarrow A \otimes A \qquad \text{coproduct}$$
$$\varepsilon : A \longrightarrow \mathbb{Z}[q, q^{-1}] \qquad \text{counit}$$
$$S : A \longrightarrow A \qquad \text{antipode}$$

Formula:
$$\Delta(K_i) = K_i \otimes K_i, \quad \varepsilon(K_i) = 1, \quad S(K_i) = K_i^{-1}$$
$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \varepsilon(E_i) = 0, \quad S(E_i) = -E_i K_i^{-1}$$
$$\Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad \varepsilon(F_i) = 0, \quad S(F_i) = -K_i F_i$$

Note: $q = 1 \rightsquigarrow K_i = 1 \rightsquigarrow$ it is a deformation of the standard coproduct on $U\mathfrak{g}$.

Need to check lots of things.

Duals of $U_q\mathfrak{sl}_2$ reps

Eg: Weyl W_3 =
$$\begin{tikzcd}
v_{-3} \arrow[r, bend left] & v_{-1} \arrow[l, bend left, "{[3]}"] \arrow[r, bend left, "{[2]}"] & v_1 \arrow[l, bend left, "{[2]}"] \arrow[r, bend left, "{[3]}"] & v_3 \arrow[l, bend left]
\end{tikzcd}$$

The dual Weyl W_3^* has basis $\{v_i^*\}$.

Have $(K v_i^*)(v_i) = v_i^*(S(K) v_i) = v_i^*(K^{-1} v_i) = v_i^*(q^{-i} v_i)$
$= q^{-i}$

$$W_3^* \qquad \begin{tikzcd} V_3^* \arrow[loop above, "q^{-3}"] \arrow[r, bend left, "-q[3]"] & V_1^* \arrow[l, bend left] \arrow[r, bend left] & V_{-1}^* \arrow[l, bend left] \arrow[r, bend left] & V_{-3}^* \arrow[l, bend left] \end{tikzcd}$$

IN GENERAL,

$$W_3^* = \begin{tikzcd} \bullet \arrow[r, bend left, "{[3]}"] & \bullet \arrow[l, bend left, "1"] \arrow[r, bend left, "{[2]}"] & \bullet \arrow[l, bend left, "{[2]}"] \arrow[r, bend left, "1"] & \bullet \arrow[l, bend left, "{[3]}"] \end{tikzcd}$$

HAVE $(E v_3^*)(v_1) = v_3^*(S(E)v_1) = v_3^*(-EK^{-1}v_1)$

$= -q\, v_3^*(E v_1) = -q[3]$

IN FACT, $W_3^* \cong W_3$ IF q GENERIC OR IF $q^{2l} = 1$, $l \geq 3$

IN GENERAL, AT $q^{2l}=1$, $W_{\lambda < l}$ SELF-DUAL, $W_{\lambda \geq l}$ NOT SELF-DUAL.

TENSOR PRODUCTS OF $U_q sl_2$ REPS

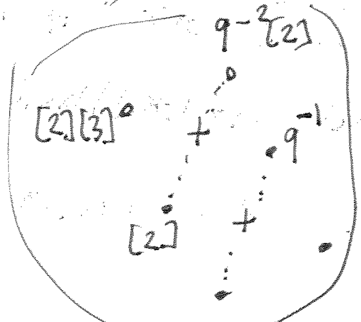
Eg: q GENERIC, $W_1 \otimes W_2$

$$\begin{tikzcd} a \arrow[r, bend left] & b \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, bend left] & B \arrow[l, bend left] \arrow[r, bend left, "{[2]}"] & C \arrow[l, bend left] \end{tikzcd}$$
 $[2]$

$$\begin{tikzcd} a\otimes A \arrow[r, "q^{-2}"] \arrow[d] & b\otimes A \arrow[d] & \\ a\otimes B \arrow[ur] \arrow[d, "{[2]}"] & b\otimes B \arrow[d, "{[2]}"] \\ a\otimes C \arrow[r, "q^2"] & b\otimes C \end{tikzcd}$$

ACTION OF E

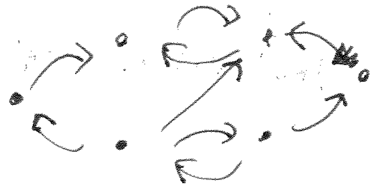
IN FACT, $W_1 \otimes W_2 \cong W_3 \oplus W_1$.



SUB-W_3

GENERATED BY $a\otimes A$ $[2][3]$

SPLITTING USES THAT q IS GENERIC (OR THAT $[3]_q \neq 0$)

Eg: $q^6 = 1$ THEN $W_1 \otimes W_2 \cong$ 

ALL COEFF ARE 1

$\rightleftarrows$ E F

DOES NOT SPLIT AS $\oplus$.

DUAL WEYL FILTRATION

BUT HAS A 2-dim SUBMODULE $\rightleftarrows$ WITH

QUOTIENT $\begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[r, bend left] & \bullet \arrow[l, bend left] \arrow[r] & \bullet \end{tikzcd}$ DUAL WEYL W_3^* AS $[3]_q = 0$

ALSO HAVE A SUBMODULE  WITH
QUOTIENT 

WEYL FILTRATION.

Def: A TILTING MODULE IS ONE WITH BOTH A WEYL AND A DUAL WEYL FILTRATION.

(AS WE WANT A TENSOR STRUCTURE, WILL WORK WITH TILTING MODULES)

THM: TENSOR PRODUCTS OF TILTING MODULES IS TILTING.

CONCLUSION: $\mathrm{Rep}^{\mathrm{tilt}} U_q\mathfrak{g}$ IS TENSOR

ISSUE: NOT SEMI-SIMPLE

EXERCISES

1) WRITE A FORMULA FOR H_i IN TERMS OF K_i AND DECIDE WHEN IT CONVERGES.
2) IN CHECKING $U_q\mathfrak{g}$ IS HOPF, SHOW IT IS ENOUGH TO CHECK HOPF COND 1 & 2 ON ALGEBRA GENERATORS.
3) AT $q^6 = 1$, SHOW $W_2 \otimes W_2 \cong W_2 \oplus M$ AND COMPUTE THE STRUCTURE OF M.