CHRIS 3

EXERCISES: 1) CHECK $U \to V$: $\mathrm{ALG} \to \mathrm{Lie}$

$A \longmapsto (A, [a,b] = ab - ba)$

2) WRITE A BASIS FOR $U\mathfrak{sl}_2$

3) CONVINCE YOURSELF THAT THE ORDERED MONOMIALS IN THE F_i's FORM A BASIS FOR M_λ.

4) PROVE THAT THE $\mathfrak{sl}_2$ VERMA MODULES HAVE EXACTLY THE GIVEN STRUCTURE.

QSR2 & 3

INTERPRET E_i AS RAISING OPERATOR:

$V \in \mathrm{Rep}(\mathfrak{g})$, $v \in V_\lambda$ i.e. $hv = H(\lambda) v$

$$H_i(E_j v) = E_j H_i v + [H_i, E_j] v$$
$$= H_i(\lambda) E_j v + a_{ij} E_j v \qquad (a_{ij} = H_i(\alpha_j))$$
$$= H_i(\lambda + \alpha_j) E_j v$$

$$\to E_j v \in V_{\lambda + \alpha_j}$$

FUNCTIONAL FORM OF SR2:

HAVE $H_i E_j = E_j (H_i + a_{ij})$

NOTE $H_i + a_{ij} = H_i(\lambda + \alpha_j) = \tau_{\alpha_j}(H_i)(\lambda)$ FOR $\tau_{\alpha_j} : \mathrm{Fun}(\Lambda) \to \mathrm{Fun}(\Lambda)$ SHIFT FUNCT.

SO $H_i E_j = E_j \tau_{\alpha_j}(H_i)$

$\Rightarrow f E_j = E_j \tau_{\alpha_j}(f)$ FOR $f \in \{H_1, \dots, H_n\}$

AND MORE GENERALLY f POLYNOMIAL ON H_i'S AND HENCE ANY FUNCTIONS ON THE H_i'S BY CONTINUITY.

$K_i \in \mathrm{Fn}(\Lambda)$, $\quad \tau_{\alpha_j}(K_i)(\lambda) = K_i(\lambda + \alpha_j) = q^{d_i H_i(\lambda + \alpha_j)}$

$$= q^{d_i a_{ij}} q^{H_i(\lambda)} = q^{d_i a_{ij}} K_i(\lambda)$$

$\leadsto$ QSR2: $\boxed{K_i E_j = q^{d_i a_{ij}} E_j K_i}$

SIMILARLY,

QSR3: $\boxed{K_i F_j = q^{-d_i a_{ij}} F_j K_i}$

<u>SUMMARY</u>: $\quad U_q\mathfrak{g} := \dfrac{\mathbb{Z}[q,q^{-1}]\langle K_1^{\pm1}, \dots, K_n^{\pm1}, E_1, \dots, E_n, F_1, \dots, F_n\rangle}{\text{QSR1}-6}$

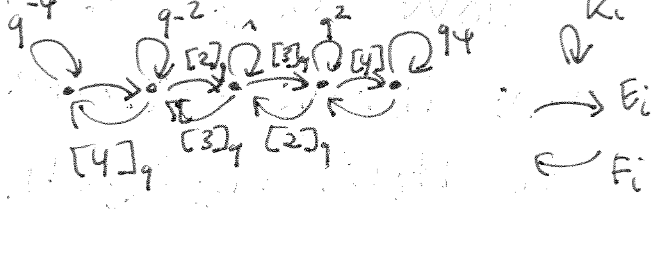
OR $\mathbb{Q}(q)$, $\mathbb{C}[q,q^{-1}]$, $\mathbb{C}(q)$

(NOTE QSR4 NEEDS A DENOMINATOR!)

<u>REPRESENTATIONS OF QG_p</u>:

REP FOR q GENERIC (EITHER OVER $\mathbb{C}(q)$, OR $q \in \mathbb{C}$, $q^\ell \neq 1$.

Eg. <u>$\mathfrak{sl}_2$</u>: $\quad U_q\mathfrak{sl}_2 \curvearrowright \mathbb{C}(q)^5$ BY



EXERCISE: CHECK IT IS A REPRESENTATION.

MINOR ANNOYANCE: $\exists$ REPRESENTATIONS THAT DON'T HAVE THAT FORM, eg

$$\begin{tikzcd} \bullet \arrow[loop left, "-q^{-1}"] \arrow[r, bend left, "-1"] & \bullet \arrow[l, bend left, "1"] \arrow[loop right, "-q"] \end{tikzcd}$$

"TYPE -1"

FIX: IGNORE THOSE! ie. ONLY STUDY "TYPE 1" REPS AS ABOVE.

CONCLUSION: $\mathrm{Rep}^{\text{TYPE }1}\left(U_q\mathfrak{g}/\mathbb{C}(q)\right) \cong \mathrm{Rep}(\mathfrak{g})$

(FOR q NOT A ROOT OF UNITY)

SPECIALIZATION:

IDEA: SET q TO BE A SPECIFIC $q \in \mathbb{C}$ IN ALL THE FORMULAS, USUALLY $q^l = 1$.

DEF: $U_q\mathfrak{g}/\mathbb{Z}[q,q^{-1}] \rightsquigarrow U_q\mathfrak{g}/\mathbb{C}(q)$

$\rightsquigarrow U_{\mathfrak{q}}\mathfrak{g} := U_q\mathfrak{g}/\mathbb{Z}[q,q^{-1}] \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C}_{\mathfrak{q}}$

SPECIFIC $\mathfrak{q} \in \mathbb{C}$ WITH $\mathfrak{q}^l = 1$ GIVING THE ACTION OF $\mathbb{Z}[q,q^{-1}]$ ON $\mathbb{C}$.

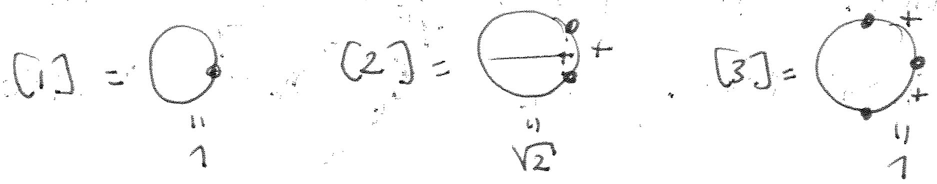
NOTE: • THESE TWO VERSIONS OF $U_q\mathfrak{g}$ ARE VERY DIFFERENT!

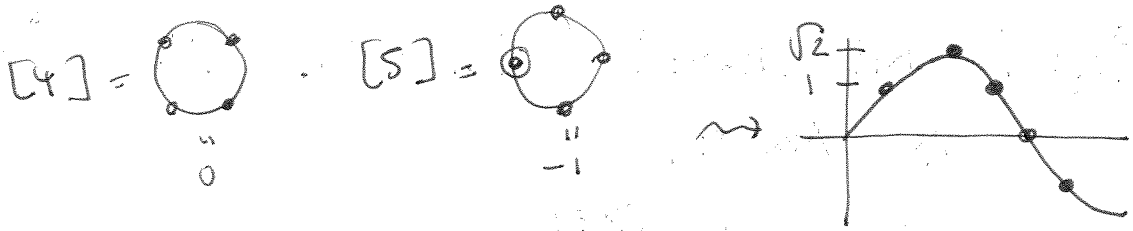
• STRUC CONTROLLED BY VANISHING

$\begin{bmatrix} n \\ r \end{bmatrix}_q$, $\dfrac{k-k^{-1}}{q-q^{-1}}$, $[n]$

CASE 1 : q GENERIC $\rightsquigarrow$ $[n]_q \neq 0$

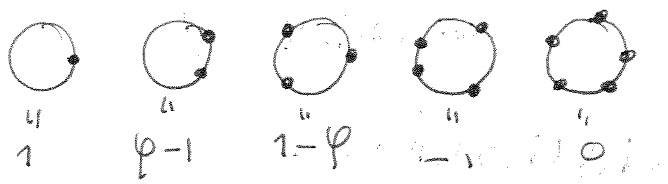
CASE 2 : q EVEN ROOT OF UNITY, eg $q^{2l}=1$, $l=4$

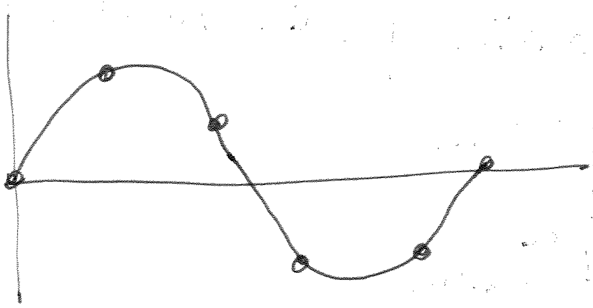




CASE 3 : q ODD ROOT OF 1, $q^l = 1$

Eg: $q^l = 1$, $l = 5$





ODD AND EVEN CASES ~~ARE~~ RATHER DIFFENT. PEOPLE USUALLY EITHER STUDY THE ONE OR THE OTHER. THE CASE HAVING THE PROPERTIES WE WANT IS THE EVEN CASE $\longrightarrow$ WE ARE IN THE "EVEN CASE".

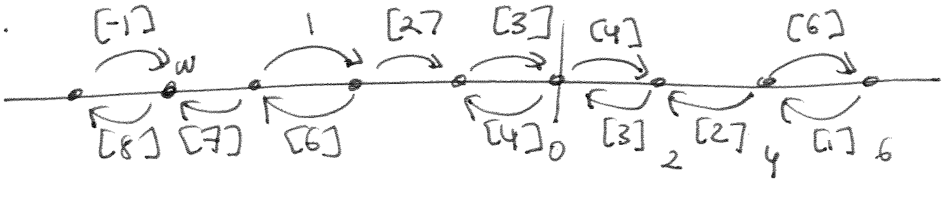
REP FOR q SPECIAL

OK WITH $U_q^{res}\mathfrak{g}$, TO BE DEFINED

DEF: VERMA MODULE ~~$M_\lambda = U_q^{res}\mathfrak{g} \otimes_{U_q\mathfrak{b}} \mathbb{C}_\lambda$~~ (LATER)

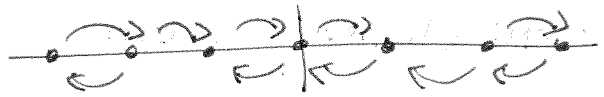
Eg sl_2, $q^{10} = 1$.

M_6



<u>DEF</u>: WEYL MODULE = (LATER)

Eg: QUOTIENT BY THE SUBMODULE GENERATED BY W (AS BEFORE)

W_6 

<u>ISSUE</u>: NOT IRREDUCIBLE (UNLIKE THE CLASSICAL CASE)

BECAUSE OF $[5]_q = 0$ IN THE PARTICULAR CASE.

<u>PLAN</u>: $\operatorname{Rep} U_q\mathfrak{g} \rightsquigarrow \operatorname{Rep} U_q\mathfrak{g}/\sim$

$W_{\lambda<\ell}$ SIMPLE

$W_{\lambda\geq\ell}$ NOT SEMI-SIMPLE

SS CAT WITH SIMPLE $W_{\lambda<\ell-1}$

SOME EQUIV. RELATION TO BE DEFINE LATER.

THE QUANTUM GROUP AS A HOPF ALGEBRA

HOPF ALGS & RIGID TENSOR CATEGORIES

<u>QUESTION</u>: WHAT STRUCTURE ON AN ALGEBRA A ENSURES THAT A-Mod IS TENSOR? (MONOIDAL)

<u>APPROACH</u>:

$$\begin{tikzcd} A\text{-mod} \otimes A\text{-mod} \arrow[r, dashed] \arrow[d] & A\text{-mod} \arrow[d] \\ \mathrm{Vect} \otimes \mathrm{Vect} \arrow[r, "\otimes"] & \mathrm{Vect} \end{tikzcd}$$

WANT $A \curvearrowright M \otimes N$ FOR M, N A-MODULES.

NEED $\varphi: A \to A \otimes A$ COPRODUCT.

CONDITIONS THE COPRODUCT NEEDS TO SATISFY:

$$\psi(a) = \sum \psi_i^1(a) \otimes \psi_i^2(a)$$

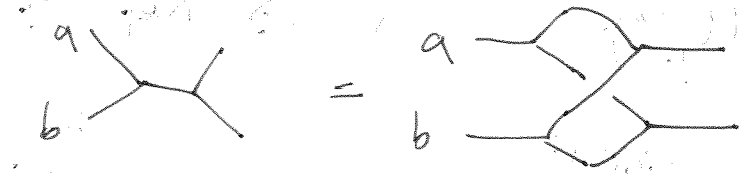
DEFINE $a.(m\otimes n) := \sum \psi_i^1(a)m \otimes \psi_i^2(a)\, n$

FOR THIS TO BE AN ACTION, NEED

$$(ab)(m\otimes n) \;=\; a(b.(m\otimes n))$$

$$\sum \psi^1(ab)m \otimes \psi^2(ab)n \qquad\qquad \sum \psi^1(a)\psi^1(b)m \otimes \psi^2(a)\psi^2(b)\, n$$

i.e. NEED



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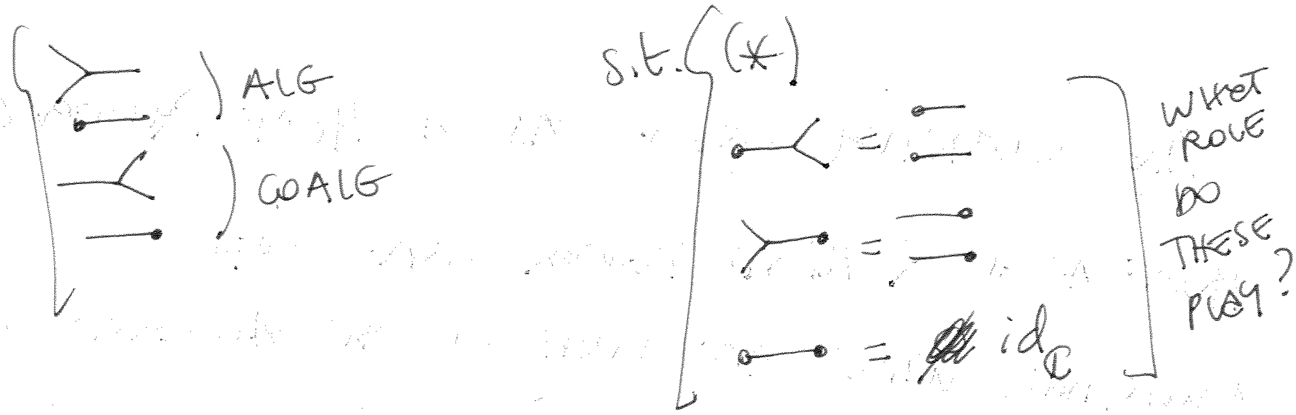
BIALG / ~~HOPF~~ CONDITION.

Def: A BIALGEBRA is



A BIALG $\Rightarrow$ A-mod is $\otimes$-CAT.