CHRIS, LECTURE 2

REPRESENTATIONS OF SIMPLE LIE ALGEBRAS

IDEA: CONSTRUCT FREE HIGHEST WEIGHT MODULES AND THEN CONSTRUCT/EXPRESS IRRED. REPS AS QUOTIENTS

VERMA MODULES: DEF: $M_\lambda := U\mathfrak{g} \otimes_{U\mathfrak{b}} \mathbb{C}_\lambda$

$\mathfrak{b} = \langle \mathfrak{h}, \mathfrak{n}_+ \rangle$
$(H_i)\ (E_i)$

$U\mathfrak{g}$: UNIVERSAL ENVELOPPING ALG = INITIAL ASSOC. ALG WITH LIE ALG MAP FROM $\mathfrak{g}$

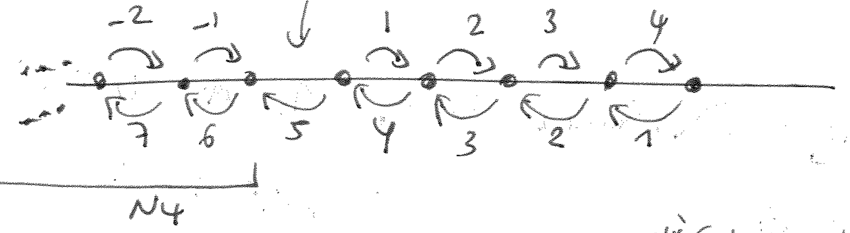
$\mathbb{C}_\lambda$: 1-dim REP, E_i TRIV. H_i ACT BY λ.

IDEA: M_λ IS FREELY GENERATED ON HIGHEST WEIGHT λ BY F_i's

$\lambda \in \Lambda = \langle H_1, \dots, H_n \rangle^*$

Eg: $\mathfrak{sl}_2$: $M_4 \cong$

GAP (E_i ACTS TRIVIALLY) ⇝ REDUCIBLE



⇝ IRREDUCIBLE REP: IDEA: IRREDUCIBLE OF HIGHEST WEIGHT λ

$$= \frac{\text{VERMA MODULE FOR } \lambda}{\text{WEIGHT SPACE FROM WHICH YOU CAN'T REACH } \lambda \text{ BY RAISING OPERATORS}}.$$

DEF: $L_\lambda := M_\lambda \Big/ \operatorname{SPAN}_{i=1,\dots,n} N_{s_i \cdot \lambda}$ WHERE N_w = SUBMODULE OF M_λ GEN. BY A VECTOR OF WEIGHT w.

SHIFTED ACTION $s_i \cdot \lambda = s_i \lambda - \alpha_i$

s_i: HYPERPLANE REFLECTION $\perp \alpha_i$

DEFORM THE TENSOR CAT OF REPS OF LIE ALG

SITUATION:

$$\begin{tikzcd} \text{LIE ALG } \mathfrak{g} \arrow[r, squiggly] \arrow[d, squiggly] & \text{CAT } \mathrm{Rep}_{\mathrm{Lie}}(\mathfrak{g}) \arrow[d, equal] \\ \text{ALG } U\mathfrak{g} \arrow[r, squiggly] & \text{CAT } \mathrm{Rep}_{\mathrm{alg}}(U\mathfrak{g}) \end{tikzcd}$$

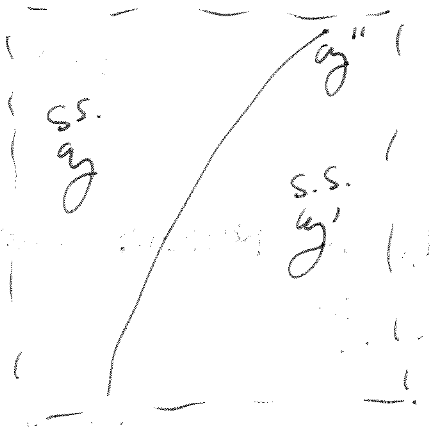
(IS HOPF) $\xrightarrow{\text{(LATER)}}$ ↗ IS A TENSOR CATEGORY

WANT TO DEFORM Rep $\mathfrak{g}$ AS A TENSOR CATEGORY.
(TO GET NEW 3-MFDS INVARIANTS)

ATTEMPT: DEFORM $\mathfrak{g}$ → FAILS! ∄ DEFORMATIONS OF SS LIE ALG'S

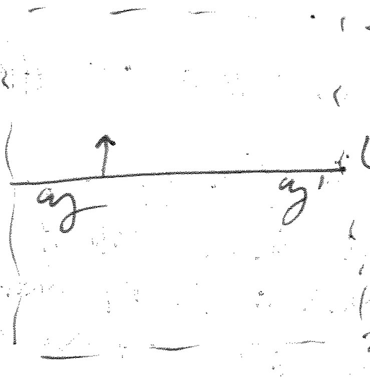
LIE ALG STRUCTURES ON $\mathbb{C}^n$:

THE SEMI-SIMPLE ALGEBRAS ARE OPEN IN THAT SPACE.



INSTEAD: DEFORM $U\mathfrak{g}$ AS A HOPF ALG.

PICTURE:



$\mathfrak{g} \leftrightarrow U(\mathfrak{g})$

→ DEFORM AWAY FROM LIE ALGEBRAS.

THE QUANTUM GROUP AS AN ALGEBRA

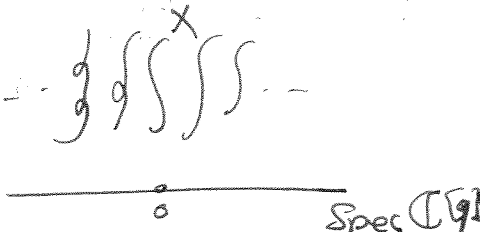
HAVE: SERRE PRESENTATION OF $\mathfrak{g}$, THUS OF $U\mathfrak{g}$, BY INTERPRETING $[a,b]$ AS $ab-ba$.

AIM: DEFORM SERRE RELATIONS.

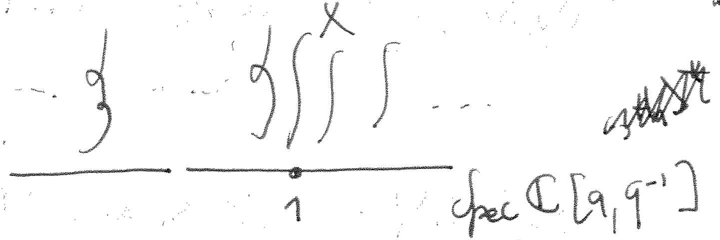
IDEA OF A DEFORMATION:

STRUCTURE ⇝ FAMILY OF SUCH STRUCTURES

eg VARIETY $X/\mathbb{C} \rightsquigarrow Y/\mathbb{C}[q]$



OR $Y/\mathbb{C}[q,q^{-1}]$



IDEA OF DEFORMATION OF $U\mathfrak{g}$:

DEFORM NUMBERS IN SERRE RELATIONS. (PRIMARILY IN THE LAST RELATIONS).

QUANTUM NUMBERS: DEF:

$$\left.\begin{aligned} [1]_q &= 1 \\ [2]_q &= q+q^{-1} \\ [3]_q &= q^2+1+q^{-2} \\ [4]_q &= q^3+q+q^{-1}+q^{-3} \end{aligned}\right] \xrightarrow{q=1} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix}$$

DEF: $[n]_q! = [1]_q [2]_q \cdots [n]_q$

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[q-r]_q!\,[r]_q!}$$

EXERCISE: CHECK

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = q^{-(n+r)} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q + q^{r} \begin{bmatrix} n-1 \\ r \end{bmatrix}_q$$

AND DRAW q-PASCAL TRIANGLE.

QUANTUM SERRE RELATIONS 5 & 6:

CLASSICAL SR 5: $ad(E_i)^{|a_{ij}|+1} E_j = 0$ (in $\mathcal{U}\mathfrak{g}$)

$$\| $$

$$\sum_{r=0}^{|a_{ij}|+1} (-1)^r \binom{|a_{ij}|+1}{r} E_i^{|a_{ij}|+1-r} E_j E_i^r$$

$\rightsquigarrow$ **QSR 5:** $$\sum_{r=0}^{|a_{ij}|+1} (-1)^r \begin{bmatrix} |a_{ij}|+1 \\ r \end{bmatrix}_{q^{d_i}} E_i^{|a_{ij}|+1-r} E_j E^r = 0$$

QSR 6: SAME WITH F_i's

QSR 4: CLASSICAL SR 4: $[E_i, F_j] = \delta_{ij} H_i$

INTERPRETATION: ~~$\forall$~~ $V \in \mathrm{Rep}\,\mathfrak{g}$, $V = \bigoplus_\lambda V_\lambda$ with $H_i \cdot v = H_i(\lambda) v$ FOR $v \in V_\lambda$

THE RELATION SAYS THAT $(\lambda \in \langle H_1 \dots H_n \rangle^*)$

$[E_i, F_i]$ ACTS BY $H_i(\lambda) \in \mathbb{Z}$ ON V_λ $\to$ DEFORM THAT.

EXPECT: "$[E_i, F_i] = \delta_{ij} [H_i]_{q^{d_i}}$"

i.e. FOR $v \in V_\lambda$, $[E_i, F_j] v = \delta_{ij} [H_i(\lambda)]_{q^{d_i}} v$

$$= q^{d_i(-H_i(\lambda)+1)} + \dots + q^{d_i(H_i(\lambda)-1)}$$

$$= \frac{q^{d_i H_i(\lambda)} - q^{-d_i H_i(\lambda)}}{q^{d_i} - q^{-d_i}}$$

GUESS: RELATION IS $\rightsquigarrow$

$$[E_i, F_j] = \delta_{ij} \left(\frac{q^{d_i H_i} - q^{-d_i H_i}}{q^{d_i} - q^{-d_i}} \right)$$

ISSUE: $\mathbb{C}[q,q^{-1}]\langle E_i, F_i, H_i \rangle / \dots$ (RELATIONS)

BUT LAST RELATION WAS ANALYTIC, NOT ALGEBRAIC !?

SOLUTION: $H_i \in \mathrm{Fun}(\Lambda, \mathbb{Z})$

$H_i \in \mathrm{Fun}(\Lambda, \mathbb{Z}[q,q^{-1}])$ CAN PICK A DIFF. BASIS FOR FUNCTIONS WITH RESP. TO WHICH THE q^H ARE ALGEBRAIC:

$K_i := q^{d_i H_i} \in \mathrm{Fun}(\Lambda, \mathbb{Z}[q,q^{-1}])$

$\rightsquigarrow$ QSR4: $\boxed{[E_i, F_j] = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}}$

QSR1: $\left\{ \begin{array}{l} \text{CLASSICAL: } [H_i, H_j] = 0 \\ [K_i, K_j] = 0 . \end{array} \right.$