QUANTUM GROUPS AND 3-MANIFOLDS

CHRIS LECTURE 1

LIE ALGEBRAS AND THEIR REPRESENTATIONS

(WARM-UP)

Def: $SL_2 = \{ M \in M_2\mathbb{C} \mid \det M = 1 \}$

Def: $sl_2 = \{ m \in M_2\mathbb{C} \mid \operatorname{tr} M = 0 \}$ = TANGENT SPACE AT Id.

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a+d=0 \right\}$$

$$= \left\langle \underset{E}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}, \underset{F}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}, \underset{H}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \right\rangle$$

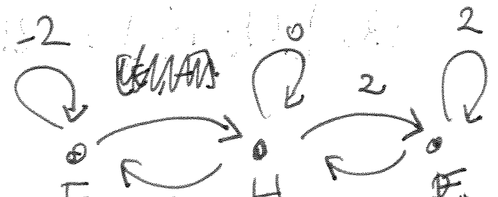
SERRE PRESENTATION: $sl_2 = \mathbb{C}\langle E, F, H \rangle$ with

$[H, E] = 2E$

$[H, F] = -2F$

$[E, F] = H$

ROOT PICTURE:



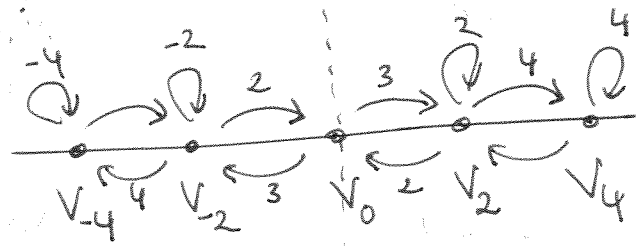
↻ $[H, -]$

→ $[E, -]$

← $[F, -]$

REPR. OF sl_2:

Eg:



L_4

ACTION OF

E →

F ←

H ↻

5-dim REP $\equiv$ LIE ALG MAP $sl_2 \to \operatorname{end}(\mathbb{C}^5)$

CLASSIFICATION: $\exists!$ IRREDUCIBLE REP OF $\mathfrak{sl}_2$ L_n FOR EACH $n \in \mathbb{N}$

EXERCISE: DRAW THE ACTION OF $\mathfrak{sl}_2$ ON L_3

$\mathfrak{sl}_3$:

$$\mathfrak{sl}_3 = \Big\langle \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{E_1}, \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ & 0 & \\ & & \end{pmatrix}}_{[E_1, E_2]}, \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{E_2}, \underbrace{F_1, [F_1, F_2], F_2}_{\text{(TRANSPOSE)}}, \underbrace{\begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & 0 \end{pmatrix}}_{H_1}, \begin{pmatrix} 0 & & 0 \\ & 1 & \\ 0 & & -1 \end{pmatrix} \Big\rangle$$

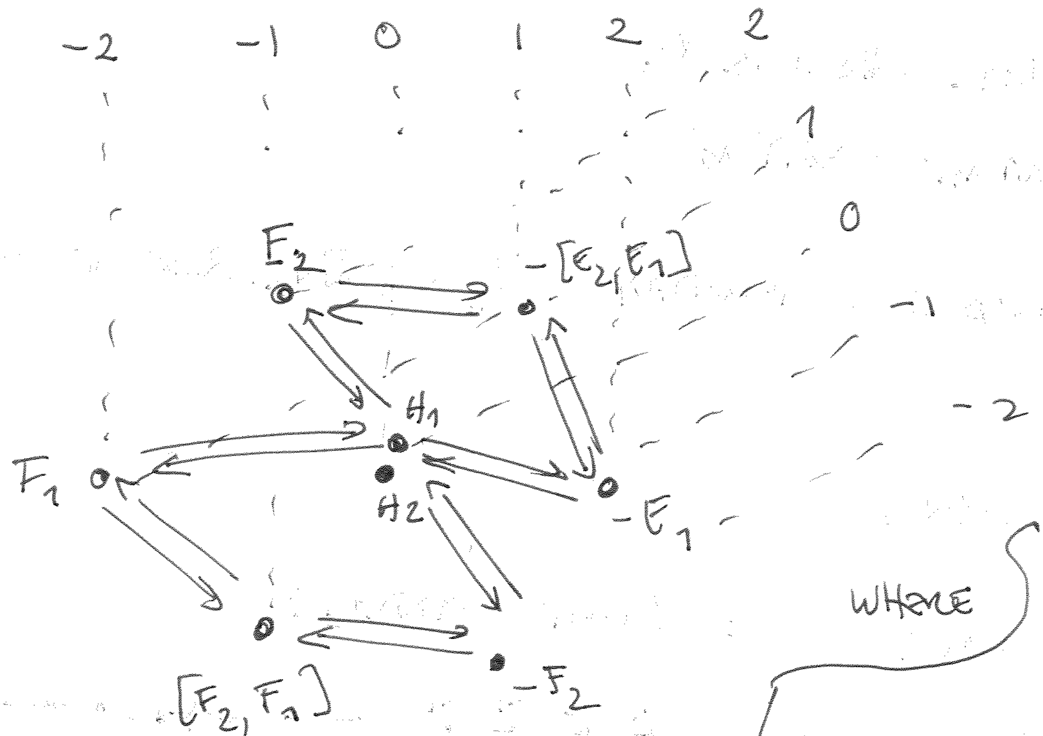
SERRE PRESENTATION: $\mathfrak{sl}_3 = \langle E_1, E_2, F_1, F_2, H_1, H_2 \rangle$

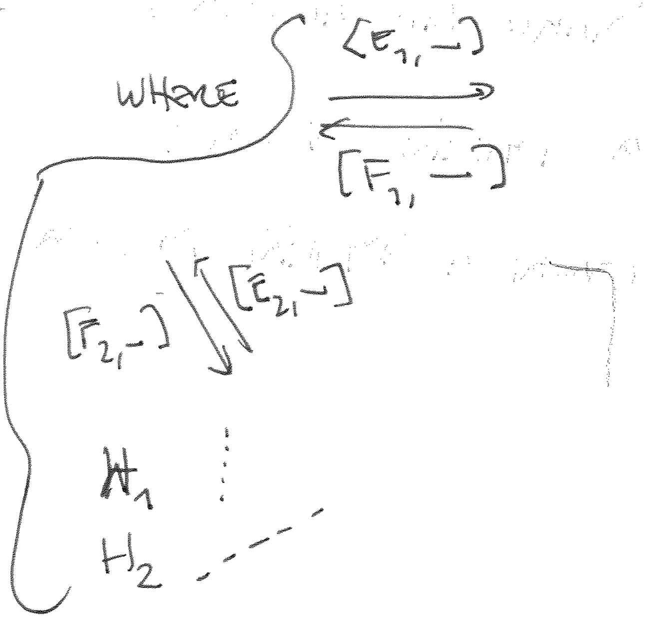
TRUE FOR $\mathfrak{sl}_2$ AND IN FACT ANY LIE ALG

1. $[H_i, H_j] = 0$
2. $[H_i, E_i] = 2E_i$ $\quad$ $[H_1, E_2] = -E_2$, $\quad [H_2, E_1] = -E_1$
3. $[H_i, F_i] = -2F_i$ $\quad$ $[H_1, F_2] = F_2$, $\quad [H_2, F_1] = F_1$
4. $[E_i, F_j] = \delta_{ij} H_i$
5. $[E_1, [E_1, E_2]] = 0$, $\quad [E_2, [E_2, E_1]] = 0$
6. SAME WITH F'S

SPECIAL FOR $\mathfrak{sl}_3$

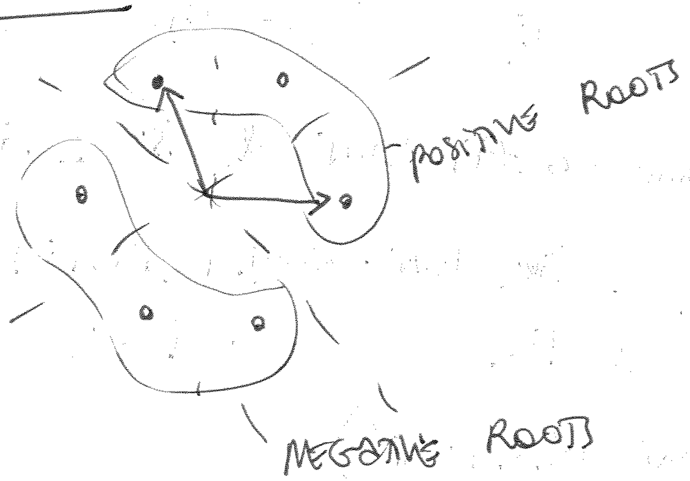
ROOT PICTURE FOR sl_3:





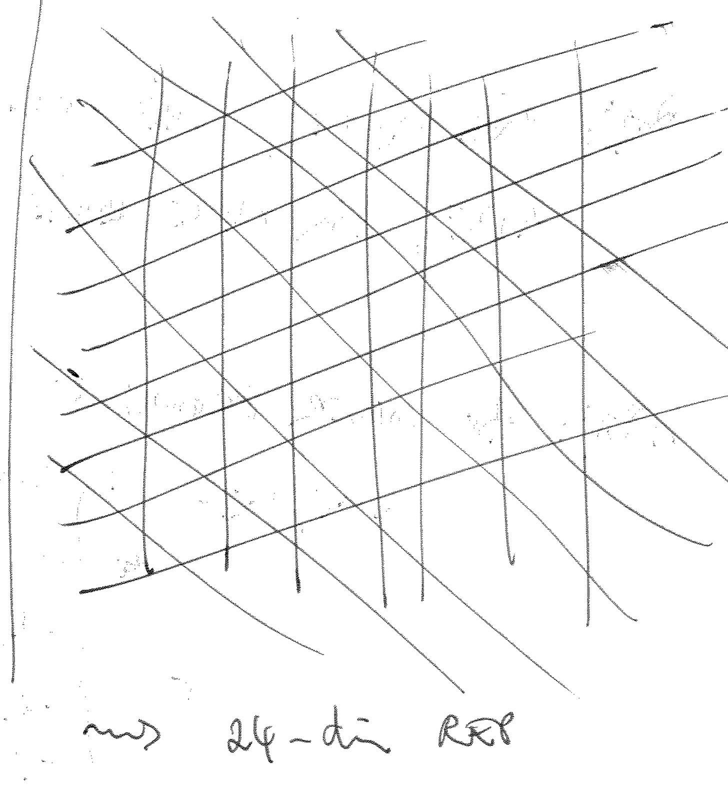
SIMPLIFY

ROOT PICTURE:



DYNKIN DIAGRAM: 

REP OF sl_3



⇝ 24-dim REP

CLASSIFICATION: $\exists!$ IRREDUCIBLE FOR EACH $\lambda \in C \cap \Lambda$

FOR C WEYL CHAMBER

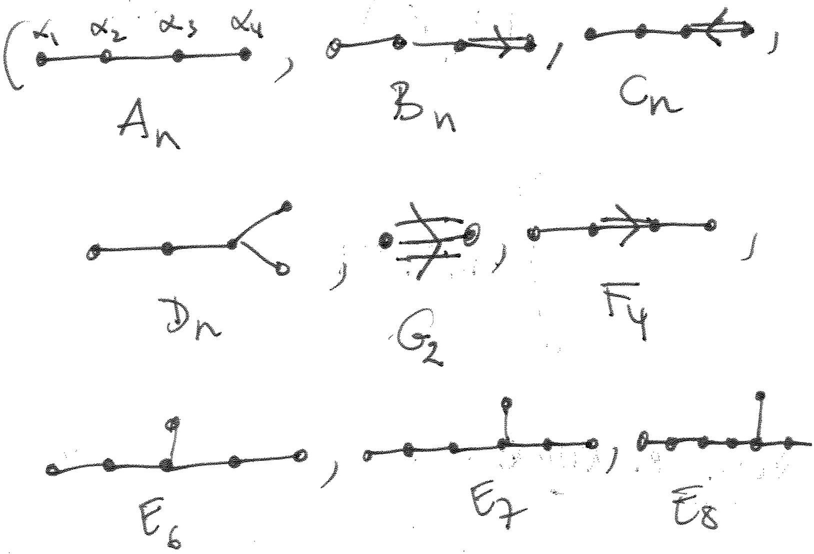
Λ WEIGHT LATTICE

EXERCISE: DRAW COMPLETELY $\mathfrak{sl}_3 \curvearrowright$ IRR. REP OF dim 3
— 6
— 10

SIMPLE LIE ALGEBRA

~~α~~ DYNKIN DIAGRAM $\rightsquigarrow$ ROOT SYSTEM:

GIVEN A DYNKIN DIAGRAM (



DEF: FOR $\alpha_i \in D$, THE LACING COEFFICIENT d_i IS 1 FOR TYPE ADE, FOR SHORT ROOTS, 2 FOR LONG ROOTS (BCF)
3 FOR ______ (G)

DEF: AN INNER PRODUCT ON $\mathbb{R}\langle\{\alpha_i\}\rangle$

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2d_i & \text{if } i = j \\ 0 & \text{if not } \overset{\text{direct}}{\text{connected}} \\ -d_i & \text{if} \\ -2 & \text{if} \\ -3 & \text{if} \end{cases}$$

THE HYPERPLANE REFLECTIONS GENERATE A FINITE ROOT SYSTEM. (SPECIAL PROPERTY OF THESE DIAGRAMS)

~~of~~ ROOT SYSTEM $\rightsquigarrow$ LIE ALG:

Def: FOR AN ORDERED PAIR α_i, α_j OF SIMPLE ROOTS, THE NILPOTENCE ORDER $a_{ij} := \dfrac{\langle \alpha_i, \alpha_j \rangle}{d_i}$. (a_{ij}) = CARTAN MATRIX

SERRE PRESENTATION: (FOR A GENERAL SIMPLE LIE ALG)

$$\mathfrak{g}_0 := \underline{\mathbb{C}\langle E_1, E_2, \ldots, E_n, F_1, \ldots, F_n, H_1, \ldots, H_n \rangle}$$

1. —
2. $[H_i, E_j] = a_{ij} E_j$
3. $[H_i, F_j] = -a_{ij} F_j$
4. —
5. $\mathrm{ad}(E_i)^{|a_{ij}|+1} E_j = 0$
6. $\mathrm{ad}(F_i)^{|a_{ij}|+1} F_j = 0$

$\Big\}\ i \neq j$

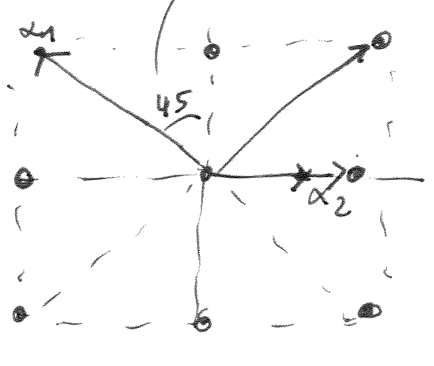
ANGLE DETERMINED BY THE NUMBER OF LINES

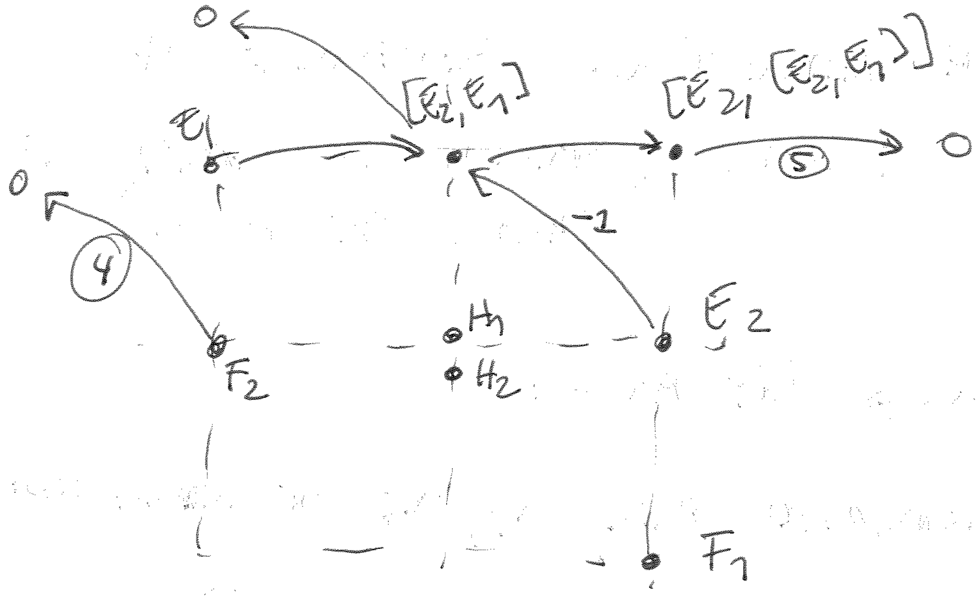
Eg: B_2 $\underset{\alpha_1}{\bullet} \Rightarrow \underset{\alpha_2}{\bullet}$

d_i: 2 1

$$\langle , \rangle = \begin{pmatrix} 4 & -2 \\ -2 & 2 \end{pmatrix}$$

$a_{12} = -1$, $a_{21} = -2$, $a_{11} = 2 = a_{22}$





EXERCICE: DRAW SAME FOR G_2.