ANDRÉ 3

3-d TQFTS

$\mathrm{Bord}_{2,3} \longrightarrow \mathrm{Vect}$

ISSUE: NEED A MANAGEABLE NBR OF GENERATORS AND RELATIONS, WHICH IS NOT THE CASE FOR $\mathrm{Bord}_{2,3}$

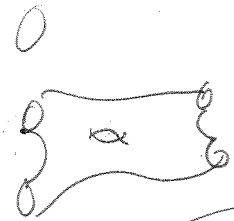
⇝ WORK WITH FUNCTORS

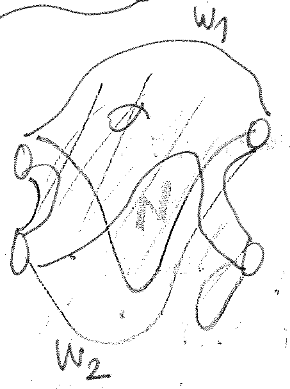
$Z : \mathrm{Bord}_{1,2,3} \longrightarrow \mathrm{Lin\,Cat}$ " = $\left\{ \begin{array}{l} \text{CATEGORIES} \\ \text{FUNCTORS} \\ \text{NAT TRANSF} \end{array} \right.$ INSTEAD.

2-CATEGORY

OR SOMETHING LIKE THE CATEGORY OF MODULES OVER AN ALGEBRA

- OBJECTS = 1-MFDS
- 1-MORPH = 2-MFDS COBORDISMS
- 2-MORPH = 3-MFDS COBORDISMS BETWEEN COBORDISMS



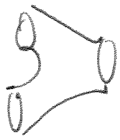


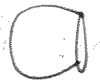
$$\begin{tikzcd} M_1 \arrow[r, bend left=50, "W_1"] \arrow[r, bend right=50, "W_2"'] \arrow[r, phantom, "\Downarrow N"] & M_2 \end{tikzcd}$$

$$\partial N = W_1 \underset{M_1 \cup M_2}{\cup} W_2$$

DEFINITION OF A SYMM. MONOIDAL BICATEGORY (2-CATEGORY): SEE APPENDIX C IN SCHOMMER-PRIES, 2d-ETFTS —

CONSIDER $Z(S^1) = \mathcal{C}$ — WHAT STRUCTURE DOES IT HAVE?

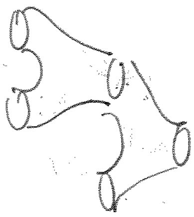
HAVE  $\rightsquigarrow$ $\mathcal{C} \otimes \mathcal{C} \xrightarrow{m} \mathcal{C}$

 $\rightsquigarrow$ $\mathrm{Vect} \longrightarrow \mathcal{C}$ (= fd VECT SPACES)

↑ UNIT IN LinCat (SEE LATER)

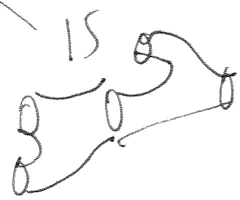
PICKS AN OBJECT IN $\mathcal{C}$ → WILL BE THE UNIT

(SPECIFIED BY $\mathbb{C} \longmapsto \mathrm{obj}\,\mathcal{C}$ AS THE FUNCTOR $\mathrm{Vect} \longrightarrow \mathcal{C}$ IS "LINEAR", IN PARTICULAR, PRESERVES $\oplus$!)



DIFFEO & COBORDISM

IS



$\rightsquigarrow$ $m(m \otimes 1)$ IS $m(1 \otimes m)$

$\rightsquigarrow$ $\mathcal{C}$ IS A TENSOR CATEGORY.

$\otimes$ OF LINEAR CATEGORIES

LINEAR CATEGORY :
- HOM SPACES ARE VECTOR SPACES
- COMPOSITION IS BILINEAR
- $\exists \oplus$ OF OBJECTS (FINITELY MANY OBJECTS)

  IN PARTICULAR, $\exists 0$ OBJECT.
  CATEGORICAL PRODUCT AND COPRODUCT

- IDEMPOTENT COMPLETE : IF $e: W \to W$, $e^2 = e$ THEN $\exists V \underset{i}{\overset{p}{\leftrightarrows}} W$ s.t. $ip = e$ AND $pi = id_V$.

TO AVOID EXAMPLES SUCH AS ALL fd VECT SPACES OF $\dim \neq 1$.

IDEMPOTENT COMPLETION: $\mathcal{C} \rightsquigarrow \hat{\mathcal{C}}$: HAS OBJECTS "$e(W)$" WITH $e : W \to W$, $e^2 = e$

AND $\mathrm{Hom}(e(W), f(V)) = f\,\mathrm{Hom}(W, V)\,e$.

EX: All ~~[illegible]~~ f.d. R-MODULES, R RING $\mathbb{Z}[x]/x^2$

NOTE: THE EXISTENCE OF $\oplus$ IN LIN CAT
$\Longleftrightarrow$ LINEAR CATEGORIES ARE Vect-MODULES
(USING $V \cong \mathbb{C} \oplus \dots \oplus \mathbb{C}$)
FOR ANY $V \in$ Vect

IDEMPOTENT COMPLETION

DEF: $\mathcal{C} \otimes \mathcal{D}$ = $\left\{ \begin{array}{l} \text{OBJECTS } \hat{\bigoplus}_{i=1} X_i \otimes Y_i \\ \mathrm{Hom}\left(\oplus X_i \otimes Y_i, \oplus X_j' \otimes Y_j'\right) \\ \quad := \bigoplus_{i,j} \mathrm{Hom}(X_i, X_j') \otimes \mathrm{Hom}(Y_i, Y_j') \end{array} \right.$

LINEAR CATEGORIES

IN VECTOR SPACES

($\rightsquigarrow$ OBJECTS ARE $e\left(\hat{\bigoplus}_{i=1} X_i \otimes Y_i\right)$ FOR $e^2 = e$ ...)

EXERCISE: $\mathrm{Vect} \otimes \mathcal{C} \cong \mathcal{C}$ (Vect = fd VECTOR SPACES)

THM: IF $\mathcal{C} = Z(S^1)$ FOR Z A 3d-TQFT, THEN $\mathcal{C}$ is semi-simple WITH FINITELY MANY OBJECTS,
i.e. $\mathcal{C} \simeq \mathrm{Vect}^{\oplus n}$

ISOMORPHISM CLASSES OF SIMPLE

1-dim ENDOMORPHISM ALG