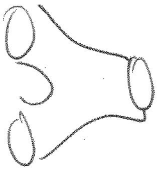
ANDRÉ 2

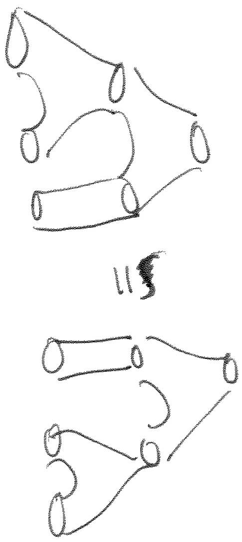
2-dim TQFT's :

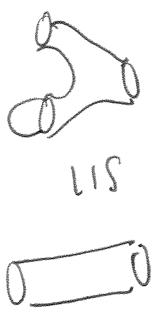
 $\longmapsto$ V

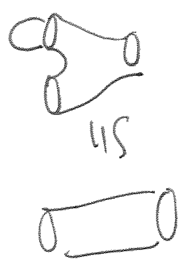
 $\longmapsto$ $[V \otimes V \xrightarrow{\cdot} V]$

 $\longmapsto$ $[\mathbb{C} \longrightarrow V]$

V UNITAL ASSOCIATIVE ALG

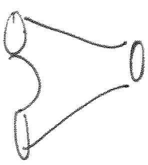
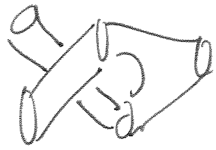






COMMUTATIVE

NOTE : " $\cong$ " = DIFFEO RELATIVE TO ∂ "
→ EQUAL MORPHISMS IN Bord.

 $\cong$ 

$a \otimes b \longmapsto ab$ $\quad$ $a \otimes b \mapsto b \otimes a \mapsto ba$

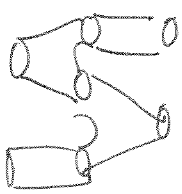
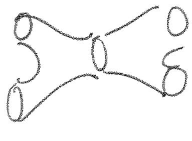
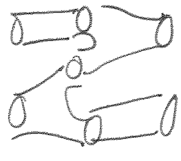
$F(S^1 \amalg S^1) = F(S^1) \otimes F(S^1)$

 $\downarrow$ $\quad$ TWIST $\downarrow \begin{smallmatrix} a\otimes b \\ \downarrow \\ b \otimes a \end{smallmatrix}$

$F(S^1 \amalg S^1) = F(S^1) \otimes F(S^1)$

$\rightsquigarrow$ COASSOCIATIVE , COCOMMUTATIVE COUNITAL COALGEBRA .

COMPATIBILITY :

 $\cong$  $\cong$ 

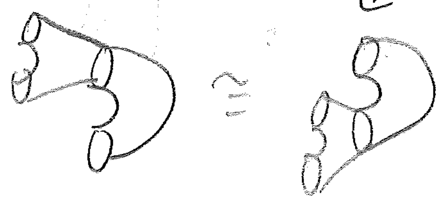
$\rightsquigarrow$ COMMUTATIVE FROBENIUS ALGEBRA .

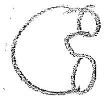
THM: $\{2\text{dim-TQFT}\} \longleftrightarrow \{\text{COMMUTATIVE FROBENIUS ALG}\}$

EXAMPLES:

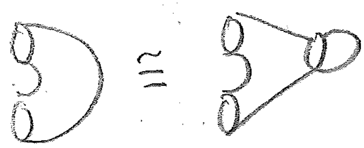
FACT: FROB ALG = UNITAL ALGEBRA V WITH $\langle \cdot , \cdot \rangle : V \otimes V \to \mathbb{C}$ NON-DEGENERATE, s.t. $\langle ab, c \rangle = \langle a, bc \rangle$

$\langle \cdot , \cdot \rangle \longleftrightarrow$ 



NON-DEG $\Rightarrow \exists$  ...

EASIEST WAY TO WRITE $\langle a, b \rangle = \varepsilon(ab)$



$\rightsquigarrow$ FROB ALG = UNITAL ALG V WITH $\varepsilon : V \to \mathbb{C}$ s.t. $\langle , \rangle = \varepsilon \circ (- \cdot -)$ IS NON-DEGENERATE.

EASIEST DEFINITION TO GET EXAMPLES.

EX: (1) $\mathbb{C}$, $\varepsilon(1) = \lambda \neq 0$. THEN $\Delta(1) = \lambda^{-1} \cdot 1 \otimes 1$
$\uparrow$ COPRODUCT.

(2) $\mathbb{C}[t]/t^{n+1}$, $\varepsilon(t^i) = \delta_{i,n}$

$\rightsquigarrow \Delta(t^i) = t^i \otimes t^n + t^{i+1} \otimes t^{n-1} + \dots + t^n \otimes t^i$.

3. $H^*(M,\mathbb{R})$, $\varepsilon = \begin{cases}\int_M - & \text{on } H^n(M) \\ 0 & \text{otherwise}\end{cases}$

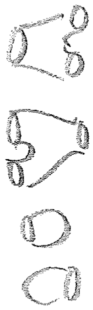
closed mfd

$\mathbb{C}$ to get an alg over $\mathbb{C}$

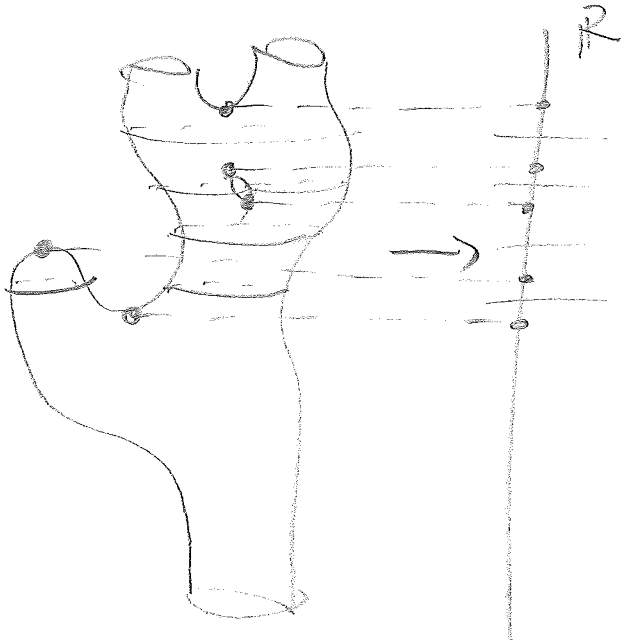
Given a commutative Frob algebra, want to construct a 2d-TQFT.

— use Morse theory.

<u>Strategy</u>: 1.



generate $\mathrm{Bord}_{1,2}$



⇝ gives a decomposition (by Morse theory) of any 2-dim mfd as a composition of the above building blocks

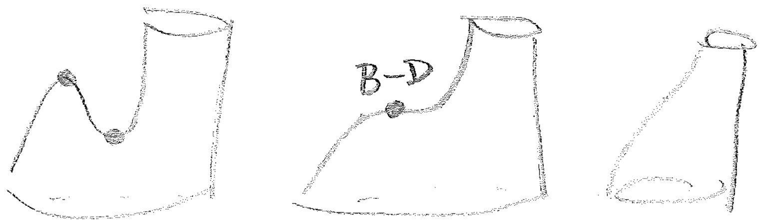
disjoint union

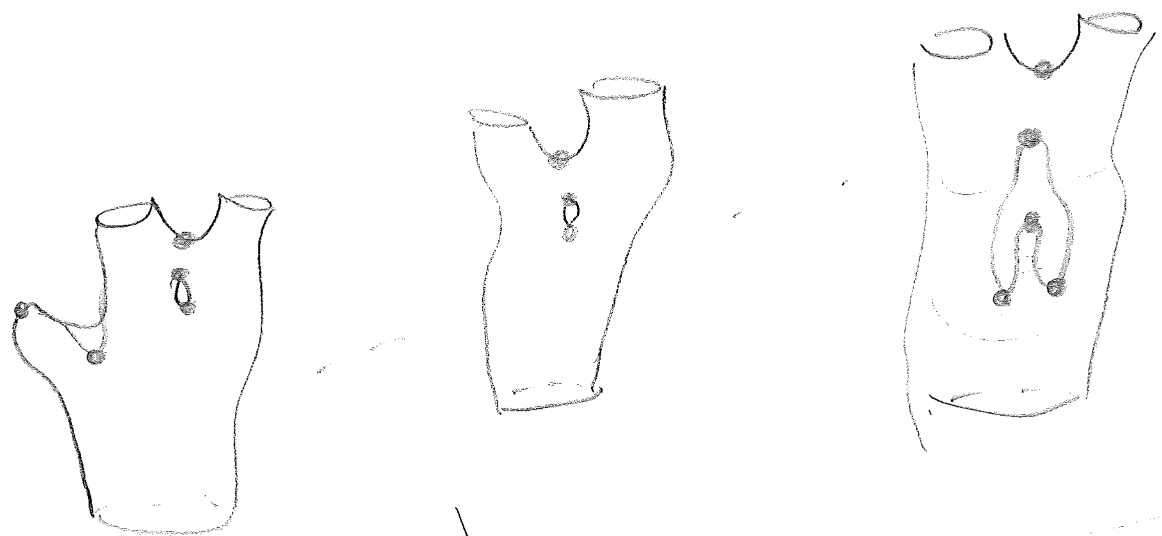
Morse function can assume one critical point per critical level

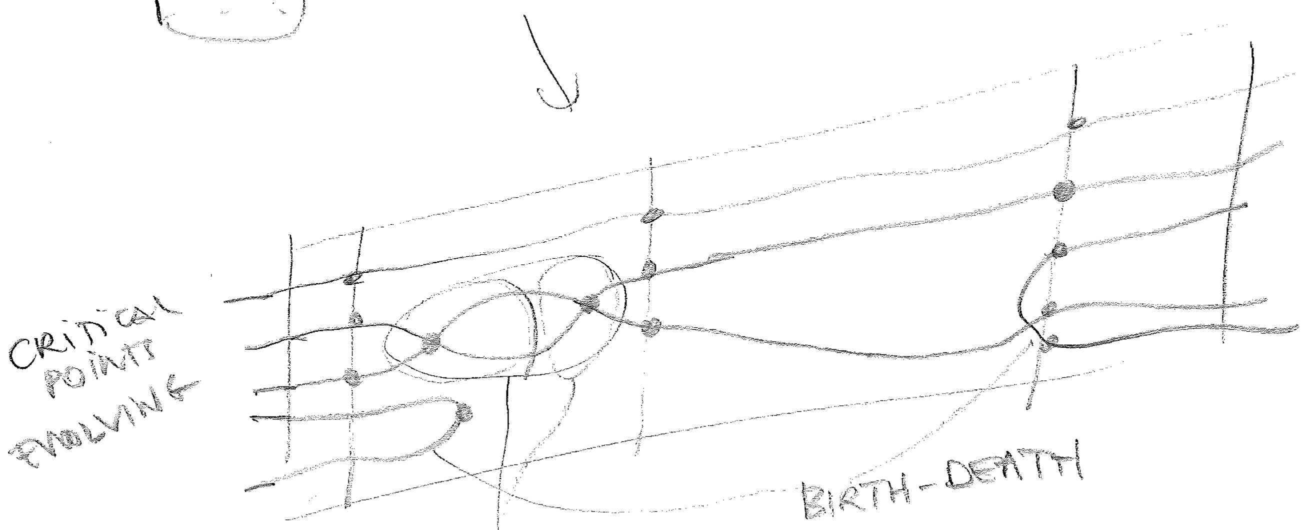
⚠ Need the multiplication and comultiplication to be (co)commutative for this to make sense.

RELATIONS:

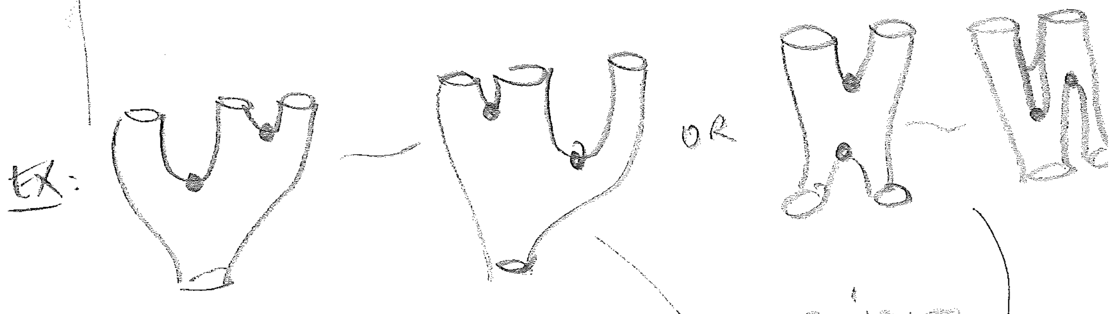
GIVEN TWO MORSE FUNCTIONS, ∃ A FAMILY OF MAPS GOING FROM THE ONE TO THE OTHER GOING THROUGH BIRTH-DEATH SINGULARITIES:



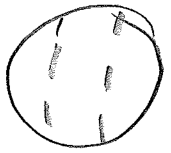


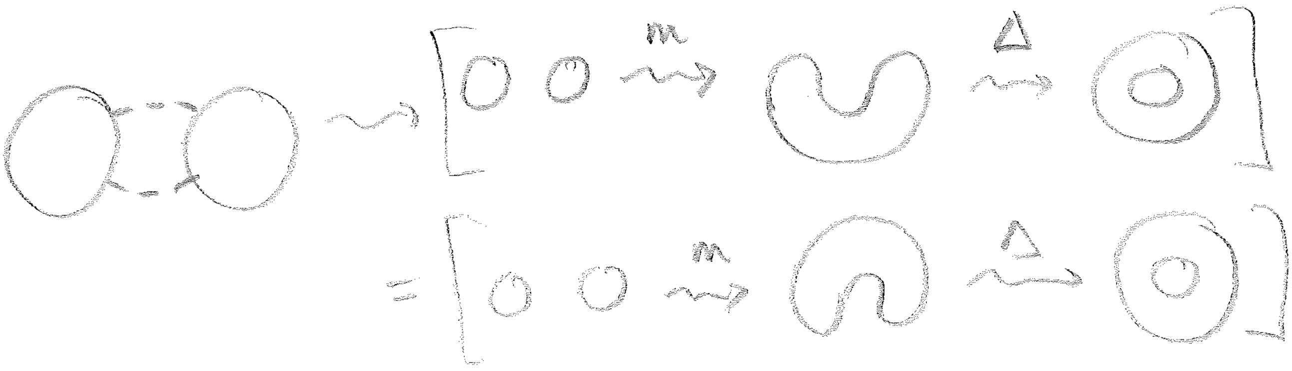


EX:

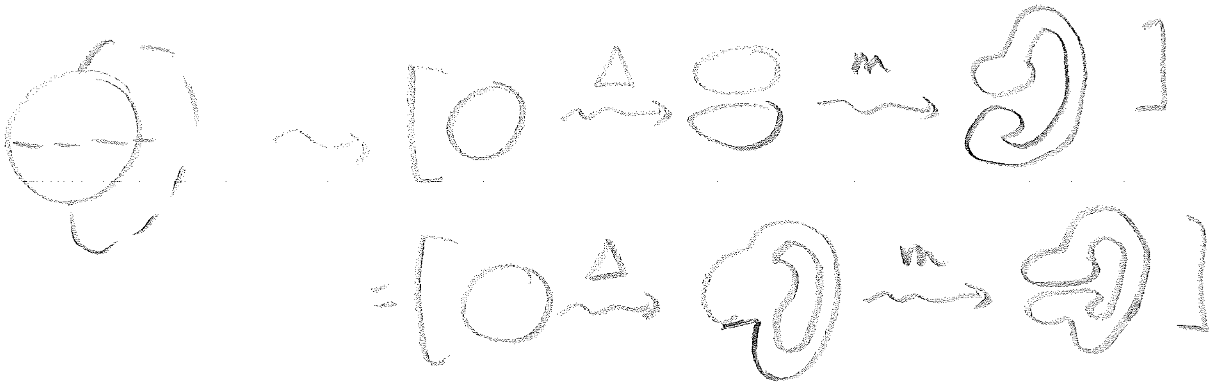


TO GET A COMPLETE LIST: WRITE ALL THE POSSIBLE COMBINATIONS OF 2 CRITICAL POINTS:

 $\rightsquigarrow$ COASSOCIATIVITY



i.e. $\Delta \circ m = \Delta \circ m$!



$m \circ \Delta = m \circ \Delta$!

$\rightarrow$ GET EXACTLY ALL THE RELATIONS OF A FROBENIUS ALGEBRA.