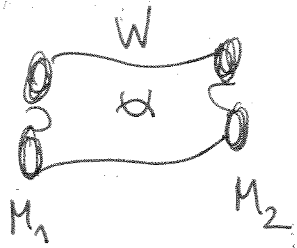
ANDRE 1

~~Def~~: A d-DIM TQFT IS A SYMMETRIC MONOIDAL FUNCTOR

$$Z : \left(\mathrm{Bord}_{d-1,d}, \amalg\right) \longrightarrow \left(\mathrm{Vect}, \otimes\right)$$

$\begin{cases} \text{OBJ} = \text{CLOSED } (d-1)\text{-MFDS} \\ \text{MORPH} = \text{EQUIVALENCE CLASSES OF COBORDISMS (UNDER DIFFEOMORPHISMS)} \end{cases}$



ORIENTED, AND $W : -M_1 \longrightarrow M_2$

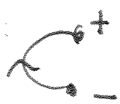
OPPOSITE ORIENTATION; SAME ORIENTATION AS W

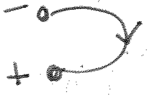
- $Z(W_1 \cup_M W_2) = Z(W_2) \circ Z(W_1)$
- $Z(M_1 \amalg M_2) = Z(M_1) \otimes Z(M_2)$, $Z(W_1 \amalg W_2) = Z(W_1) \otimes Z(W_2)$
- $Z(\emptyset) = \mathbb{C}$

NOTE: $Z(\emptyset \xrightarrow{W} \emptyset) = (\mathbb{C} \xrightarrow{Z(W)} \mathbb{C})$ $\rightsquigarrow$ COMPLEX NBR ASSOCIATED TO ANY CLOSED d-MFD.

EXAMPLE $d=1$:

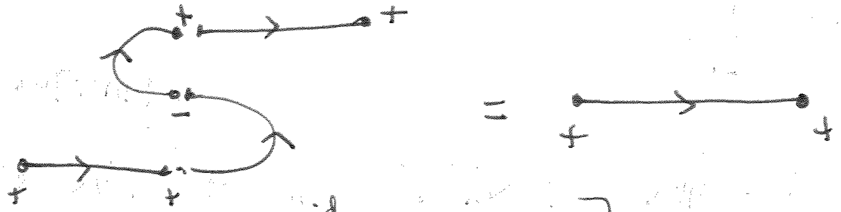
GIVEN A 1-dim TQFT, WE GET:

$\cdot^{+} \longmapsto V$  $\longmapsto [\mathbb{C} \xrightarrow{\mathrm{coev}} V \otimes W]$

$\cdot_{-} \longmapsto W$  $\longmapsto [W \otimes V \xrightarrow{\mathrm{ev}} \mathbb{C}]$

($\exists$ TWO ORIENTED CONNECTED 0-MFDS) GENERATE THE WHOLE FUNCTOR.

RELATIONS:



(*)

$$\left[V = \mathbb{C}\otimes V \xrightarrow{\mathrm{coev}\,\otimes\,\mathrm{id}} V\otimes W\otimes V \xrightarrow{\mathrm{id}\,\otimes\,\mathrm{ev}} V\otimes \mathbb{C} = V \right] = V \xrightarrow{\mathrm{id}} V$$

LIKEWISE

(**)

<u>Def</u>: $\mathcal{C}$ MONOIDAL CATEGORY

$V \in \mathcal{C}$ OBJECT.

A RIGHT <u>DUAL</u> TO V IS A TRIPLE $(W, \mathrm{ev}, \mathrm{coev})$ (AS ABOVE) SATISFYING (*) AND (**).

(AND V IS THE <u>LEFT DUAL</u> OF W).

IF $\mathcal{C}$ IS SYM. MONOIDAL, LEFT = RIGHT FOR DUALS.

SUPPOSE $(W', \mathrm{ev}', \mathrm{coev}')$ IS ANOTHER RIGHT DUAL OF V,

THEN

$$(\mathrm{ev}\otimes 1)\circ(1\otimes \mathrm{coev}') : W \longrightarrow W'$$

$$(\mathrm{ev}'\otimes 1)\circ(1\otimes \mathrm{coev}) : W' \longrightarrow W$$

ARE INVERSE ISOMORPHISMS.

$\rightsquigarrow$ CANONICAL ISOMORPHISM $W \xrightarrow{\cong} W'$.

USE THE NOTATION V^* FOR W.

EXAMPLE: $V \in \mathrm{Vect}$, $V^* = \mathrm{Hom}(V, \mathbb{C})$ DUAL VECTOR SPACE.

$V^* \otimes V \xrightarrow{ev} \mathbb{C}$ IS THE EVALUATION.

$$\mathbb{C} \xrightarrow{coev} V \otimes V^*$$

$$1 \longmapsto \sum_{i=1}^{n} e_i \otimes e^i$$

FOR e_i A PICKED BASIS FOR V, $\{e^i\}$ = DUAL BASIS

MAKE SENSE IF V FIN. DIM

EXERCISE: CHECK (*) AND (**)

NOTE: COEV DOES NOT DEPEND ON THE CHOICE OF BASIS: SUPPOSE $\tilde{e}_i = a_i^j e_j$. LET $b_i^j a_k^j = \delta_i^k$ ← INVERSE MATRIX.

CLAIM: $\tilde{e}^j = b_i^j e^i$

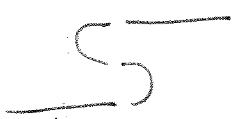
PF: $\tilde{e}^k(\tilde{e}_j) = b_l^k e^l (a_j^i e_i) = b_i^k a_j^i = \delta_j^k$!

δ_i^l

...

NOTE: IF V IS NOT FINITE DIMENSIONAL, IT DOES NOT ADMIT A DUAL.

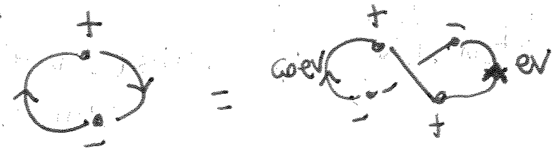
PF:



$$v \longmapsto \sum_{i=1}^{n} \underbrace{c_i \otimes d_i}_{coev(1)} \otimes v \longmapsto \sum_{i=1}^{n} c_i \cdot \underbrace{ev(d_i \otimes v)}_{\in \mathbb{C}}$$

$\Rightarrow$ $V = \mathrm{span}\langle c_i \rangle$ $\Rightarrow$ V f.d.

NOTE:



$\leftrightarrow \quad 1 \mapsto \sum e_i \otimes e^i \mapsto \sum e^i \otimes e_i$

$\mapsto \sum e^i(e_i) = \sum 1$

$= \dim V.$

$\rightsquigarrow$ 1-1 CORRESPONDENCE

$$\{\text{f.d. VECTOR SPACES}\} \longleftrightarrow \{\text{1-dim TQFT's}\}$$

STRING DIAGRAMS:

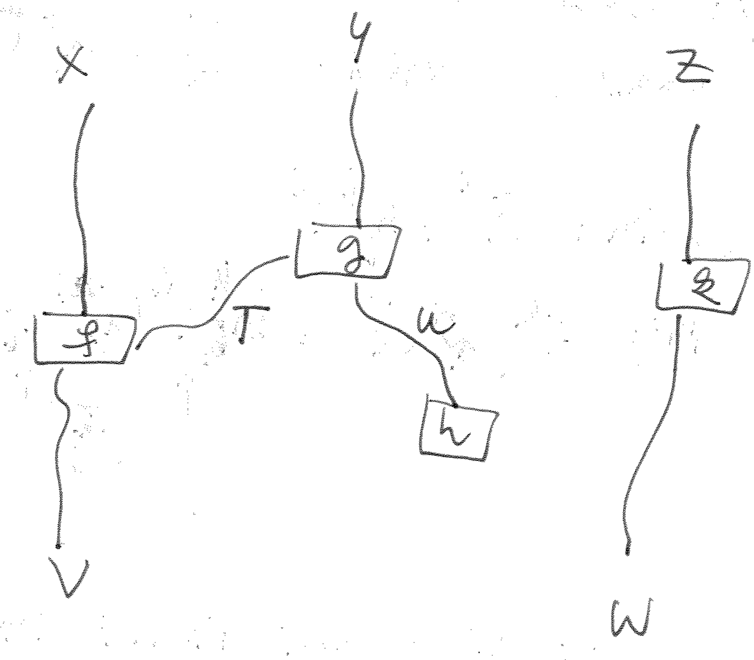
X, Y, Z, T, U, V, W OBJECTS

IN A [redacted] MON. CAT

$f: X \otimes T \longrightarrow V$

$h: U \longrightarrow \mathbb{1}$

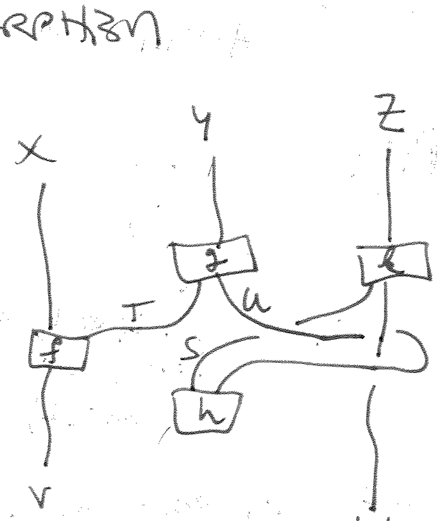
$\vdots$



$\rightsquigarrow$ DIAGRAM IS A WELL-DEFINED MORPHISM

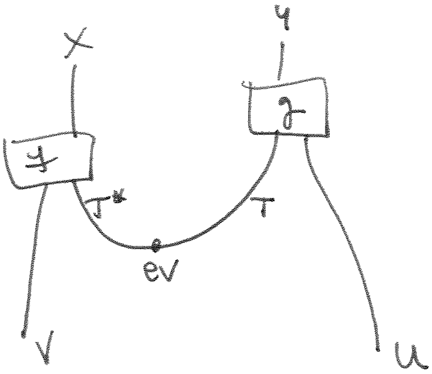
$$X \otimes Y \otimes Z \longrightarrow V \otimes W$$

IF THE CATEGORY IS BRAIDED, CAN HAVE DIAGRAMS OF THE FORM:



IF SYMMETRIC, DRAW  INSTEAD OF  OR .

IF OBJECTS HAVE DUALS, ALLOW



IF DUALS AND BRAIDING EXIST,