Q&A Session

$\boxed{\text{Q1}}$: (concerning the first ~~quest~~ lecture by H.P.)
Does the construction of the Lie algebroid of a Lie groupoid works if one uses t instead of s, e.g. use T^tG instead of T^sG.

A1: Yes, one will get isomorphic Lie algebroids. The literature is divided, so one has to be careful.

$\widetilde{A2}$: There may be a canonical way of definition using the normal bundle of u. Probably a choice is still required (think of the case of Lie groups and left vs. right invariant vector fields).

$\boxed{\text{Q2}}$: Does the following groupoid play a role anywhere?

$$\begin{array}{c} E \\ \downarrow \\ M \end{array} \quad \text{vector bundle}$$

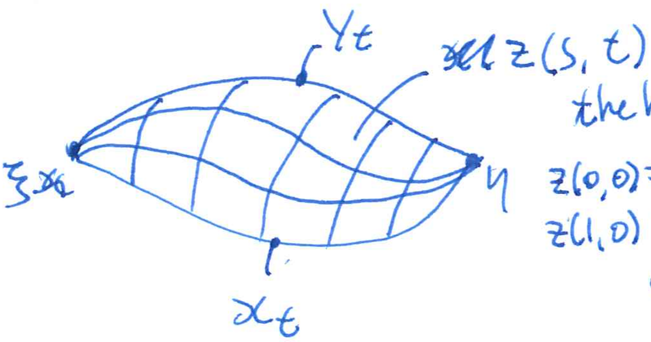
Objects: $x \in M$

Morphisms:

$G_{x,y} = \{$ Homotopy classes of paths 

equiped with paths ~~of~~ in $GL(\text{rank}\,E)$ $\quad g_t : E_x \xleftarrow{\sim} E_{x_t} \quad g_0 = Id \}$

where homotopy means



the homotopy
$z(0,0) = x_0 = \xi$
$z(1,0) = y_0 = \eta$
etc.

and $g_t : E_\xi \xleftarrow{\sim} E_{x_t}$

$h_t : E_\xi \xleftarrow{\sim} E_{y_t}$

$I_{s,t} : E_\xi \leftarrow E_{z(s,t)}$

So
$$\begin{tikzcd} \widetilde{\mathrm{Aut}}(E_x) \arrow[r, hook] & \mathcal{G}(E) \arrow[r] & \pi_1(M) \arrow[l, bend right, dashed] \\ & (\gamma, g_t) \arrow[r, mapsto] & [\gamma] \end{tikzcd}$$

the constructed groupoid

The - - - section would be given by a flat connection on E.

A: Maybe, it has not been seen by anyone in the room so far.

$\boxed{Q3}$: Another question in the same vein

$G \rightsquigarrow dg \cdot g^{-1} \in \Omega^1(G, \mathfrak{g})$ Maurer-Cartan form

A: There is something like this.

$\boxed{Q4}$: ~~Ho~~ (Now concerning the first lecture by B. T.)

How is the discussion on Kernels ~~related~~ in index theory a motivation for looking at $\mathbb{A}(M)$ modules.

A: Let's see

$$K_f(x,y) = \int e^{\frac{(x-y)\xi}{i\hbar}} f(x,\xi)\, d\xi$$

$\hbar$-dependent (generalized) function in x and y.

Recall that $\mathbb{A}(\mathbb{R}^{2n})$ came from $\underbrace{\text{differential operators}}_{\text{act on (generalized) functio}}$

Example: usual functions

$$\left(\sum P_k(x)\cdot \xi^k\right) a(x) = \sum P_k(x)(i\hbar\partial_x)^k a(x)$$

Note: we can write $F\cdot a = \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!}\partial_\xi^k F\cdot \partial_x^k a\Big|_{\xi=0}$

$F\in A(\mathbb{R}^2)$ $a\in C^\infty(\mathbb{R})[[\hbar]]$ $F\cdot a$ action.

So a module supported at $\{\xi=0\}$

Example 2: - extra parameter θ
- function $\varphi(\underline{x},\theta)$

(Ex: $\underline{x}=(x,y)$
$\xi=\theta$
$\varphi(\underline{x},\theta)=(x-y)\xi$)

Consider generalized functions of the form:

$$\int e^{\frac{\varphi(\underline{x},\theta)}{i\hbar}} \underbrace{a(\underline{x},\theta)}_{\text{smooth}} d\theta = u(x)$$

Ex 2.1 $\theta=\{0\}$ $u(\underline{x}) = e^{\frac{\varphi(\underline{x})}{i\hbar}} a(\underline{x})$

$$\xi\cdot u = i\hbar\frac{\partial}{\partial x} e^{\frac{\varphi}{i\hbar}}\cdot a = \varphi' e^{\frac{\varphi}{i\hbar}} a + e^{\frac{\varphi}{i\hbar}} i\hbar\partial_x a$$

So $\xi: a\longmapsto i\hbar\partial_x a + \varphi' a$ Note: $(\xi-\varphi'(x)): 1\longrightarrow 0$

$x: a\longmapsto a$

then $F(x,\xi)\underset{\text{New}}{\bullet} a(x) = F(x,\xi-\varphi'(x))\underset{\text{old}}{\bullet} a(x)$

Conclusion: New module of $A(\mathbb{R}^2)$ Supported on $\xi=\varphi'(x)$

More on this in the next lecture.

$\boxed{Q5}$: 2 in the outline concer~~d~~ned classification of Deformation Quantizations of a given Poisson manifold $(X, \{\cdot, \cdot\})$. What about classification of the Poisson structure, how much is known?

A: Not very much, it is definitely beyond the scope of the lectures to go into it. Poisson Structures are very ~~wa~~ abundant. One can try to restrict the class of Poisson brackets one wants to ~~stud~~ classify. ~~#~~ Like only look at those which share a certain singular locus. To see the abundance of such structures just note that $\forall f \in C^\infty(\mathbb{R}^2)$ $f\pi$ is a not necessarily isomorphic Poisson structure on $\mathbb{R}^2$, given the standard structure π. For instance if f vanishes anywhere ~~π~~ $f\pi \not\simeq \pi$.

References: Bursztyn ⊢ Radko — on classification of a class of Poisson structures

Marcut ⊢ on the moduli space of a sphere in $\mathfrak{g}^*$ for $\mathfrak{g}$ a compact semi-simple Lie algebra.

$\boxed{Q6}$: IS every Lie Groupoid weakly equivalent to a Lie group acting on a manifold?

A: This is actually an open question.

Q'er: Yes!

A: It is definitely not true in general, but what conditions are necessary and sufficient is unknown. It is true for proper étale groupoids.

For proper Lie groupoids there is a slice theorem that says that locally it looks like a compact Lie group acting on an open subset of $\mathbb{R}^n$. "if $G \rightrightarrows M$ a Lie Groupoid then it is proper if $(s,t): G \rightrightarrows M \times M$" is proper.

Slice theorem (Zung) (See above).

$\boxed{Q7}$: There was an example of Lie Groupoid give by $\coprod_{i,j} U_i \cap U_j \longrightarrow \coprod_i U_i$, this look somehow like Čech homology, is it related?

A: Yes, this is sometimes called the Čech groupoid. It's groupoid cohomology can calculate the Čech cohomology of the manifold.

$\boxed{Q8}$: What do Crainic-Fernandes obstructions look like?

A: ~~[illegible]~~ There is a topological construction yielding the groupoid as the reduction of an ∞-dim groupoid (some manifold of homotopy classes of paths). If the quotient is well-defined it is the Lie groupoid. Otherwise there is an obstruction to the existence of a smooth structure.

$\boxed{Q9}$: Is it easy to see that $\mathcal{H}ol(\mathcal{F})$ and $Mon(\mathcal{F})$ are smooth?

A: No.