Lie Groupoids & Lie Algebroids

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Lie Groupoids

- group actions, foliations, differentiable stacks

  equivalence classes of Lie groupoids

- model of singular spaces
- models for NCG (convolution algebras).

Definition:

A groupoid is a small category in which every arrow is invertible.

M, set of objects

G, set of morphisms

$g \in G$, $y \xleftarrow{g} x$, $x = s(g)$, $y = t(g)$
source target

$\forall x \in M$, $\exists$
$$\begin{tikzcd} x \arrow[loop right, "1_x"] \end{tikzcd}$$

composition:
$$\begin{tikzcd} y & x \arrow[l, "g_1"'] & z \arrow[l, "g_2"'] \arrow[ll, bend left, "g_1 g_2"] \end{tikzcd}$$
if $s(g_1) = t(g_2)$

$\forall g \in G$, $\exists g^{-1}$ s.t. $gg^{-1} = 1$.

Let $G_2 = \{(g_1, g_2) \mid s(g_1) = t(g_2)\}$.
composable arrows

Then $G_2 \xrightarrow{m} G$
multiplication

$$\begin{tikzcd} G_2 \arrow[r, "m"] & G \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] \arrow[loop below, "\text{inverse } I"] & M \arrow[l, bend right, "u \text{ unit}"'] \end{tikzcd}$$

These maps should satisfy:

(1) associativity of m (whenever possible)
(2) identity axioms for units.

<u>Definition</u>:

A groupoid $G \rightrightarrows M$ is a <u>Lie groupoid</u> w/ M, G smooth manifolds, $s: G \longrightarrow M$ a smooth submersion and all other maps are smooth maps.

<u>Remark</u>:

(1) s smooth submersion $\Longrightarrow$ G_2 smooth manifold
(2) $t = s \circ I \Rightarrow t$ is also a smooth submersion.

<u>Examples</u>

(1) M smooth manifold is a Lie groupoid w/ only unit arrows.

(2) <u>Pair groupoid</u>.

M smooth manifold

$M \times M \overset{s}{\underset{t}{\rightrightarrows}} M$, s.t corresponding projections

product $(x, y)(y, z) = (x, z)$

(3) $f: Z \longrightarrow M$ surjective submersion

$Z \times_M Z \rightrightarrows Z$ subgroupoid of the pair groupoid of Z

(4) $\mathcal{U} = \{U_i\}_{I \ni i}$ open cover of M, $\coprod_{i \in I} U_i \longrightarrow M$.

Applying (3) we get $\coprod_{i,j} U_i \cap U_j \rightrightarrows \coprod_i U_i$

(5) K Lie group. Then $K \rightrightarrows *$ is a Lie groupoid.

(6) $K \curvearrowright M$, M manifold. Then

$$K \times M \rightrightarrows M \qquad \begin{aligned} s(k,x) &= x \\ t(k,x) &= kx \end{aligned}$$

$$(k_1, m_1)(k_2, m_2) = (k_1 k_2, x_2)$$

(7) <u>Foliation groupoids.</u>

$F \subseteq TM$ integrable subbundle

Frobenius Thm $\Longrightarrow$ M is partitioned into leaves L.

<u>Monodromy groupoid.</u> $Mon(F) \rightrightarrows M$

$\forall x, y \in M$, an arrow between x, y is given by homotopy classes of paths in the leaves L of the foliation. This means that there are no arrows between x, y if they belong to different leaves.

<u>Holonomy groupoid.</u> $Hol(F) \rightrightarrows M$

$\forall x, y \in M$, an arrow between x, y is given by holonomy classes of paths in the leaves.

<u>Holonomy:</u>

L leaf of a foliation, $\gamma : [0,1] \longrightarrow L$ a path, $\gamma(0) = x$, $\gamma(1) = y$. Choose transversals T_x, T_y to the foliation near x, y. The holonomy of γ is a germ of a diffeomorphism

$$(T_x, x) \xrightarrow{hol(\gamma)} (T_y, y)$$

Another way to describe the foliation is by giving a foliation atlas of M. Let $M = \bigcup_i U_i$, charts

$$\varphi : U_i \longrightarrow \mathbb{R}^{n-q} \times \mathbb{R}^q$$

$n = \dim M$, $q = \operatorname{corank} F$

w/ transition functions $\varphi_{ij}(x,y) = (g_{ij}(x,y), h_{ij}(y))$.
these transition functions preserves the leaves $\varphi_i^{-1}(\mathbb{R}^{n-q} \times \{y\})$.

<u>Remarks</u>: homotopic paths have the same holonomy. This gives us a morphism between the monodromy and the holonomy groupoids

$$\begin{tikzcd} \mathrm{Mon}(\mathcal{F}) \arrow[rr] \arrow[dr, shift left] \arrow[dr, shift right] & & \mathcal{H}ol(\mathcal{F}) \arrow[dl, shift left] \arrow[dl, shift right] \\ & M & \end{tikzcd}$$

Let $G \rightrightarrows M$ be a Lie groupoid. This induces an equivalence relation on M where $x \sim y \iff \exists g \in G$ s.t. $y \xleftarrow{g} x$.

$M/\sim$ "quotient space" or "coarse moduli space"

<u>Actions of Lie groupoids</u>

An action of a Lie groupoid $G \rightrightarrows M$ on a manifold Z is given by a map $\mu : Z \to M$ together with another map

$$Z \,{}^{\mu}\!\times_M^{t} G \longrightarrow Z$$
$$(z,g) \longmapsto zg$$

satisfying $(zg_1)g_2 = z(g_1g_2)$ (whenever defined).

<u>Example of an action</u>

Lie groupoid acting on itself.

Take $Z = G$, $\mu = s : G \to M$. Then $G \,{}^{s}\!\times_M^{t} G = G_2$.

We can take the action to be the multiplication map $G \,{}^{s}\!\times_M^{t} G \xrightarrow{m} G$.

Another example:

Consider $T^sG := \ker ds \subseteq TG$. Let $A = T^sG|_M$ a vector bundle over M.

Lemma: G acts on T^sG and $T^sG/_G \cong A$.

Proof: Let $y \xleftarrow{g} x$ and consider $R_g : s^{-1}(y) \longrightarrow s^{-1}(x)$.
$h \longmapsto hg$

Note that R_g is a diffeomorphism.
By the submersion theorem, $(\ker ds)_h \cong T_h(s^{-1}(sh))$.
This isomorphism gives us

$$dR_g : T^sG \longrightarrow T^sG$$

This defines the action on T^sG. This action lifts the right action of G on itself.

by the action of G w/ itself

We have $G/_G \cong M$ via the maps $G \underset{u}{\overset{t}{\rightleftarrows}} M$.

By the lifted action, we get $T^sG/_G \cong A$. □

$\mathfrak{X}^s_{inv}(G) = \{$G-invariant vector fields on G tangent to the fibers of $s : G \longrightarrow M\}$.

Lemma: ① $\mathfrak{X}^s_{inv}(G) \leq \mathfrak{X}(G)$ as Lie algebras

② all vector fields in $\mathfrak{X}^s_{inv}(G)$ are projectable along $t : G \longrightarrow M$ to M.

Lie algebroids

M manifold. A Lie algebroid over M is a vector bundle $A \longrightarrow M$ together w/ a Lie structure on $\Gamma(A)$ and a vector bundle map $\rho : A \longrightarrow TM$ s.t. ⓐ $\rho[X,Y] = [\rho(X), \rho(Y)]$, $\forall X, Y \in \Gamma(A)$ ⓑ $[X, fY] = f[X,Y] + (\rho(X)f)Y$, $f \in C^\infty(M)$.

Remark:

Associated to a Lie groupoid is a Lie algebroid:

$$G \rightrightarrows M \rightsquigarrow A = (\ker ds)\big|_M , \ \rho = dt\big|_M .$$

Examples:

(1) M manifold w/ only unit arrows.
This has Lie algebroid $\{0\}$.

(2) $M \times M \rightrightarrows M$ pair groupoid.
$(x,y)(y,z) = (x,z)$

$$\Delta : M \longrightarrow M \times M$$
$$x \longmapsto (x,x) \quad \text{(unit map = diagonal)}$$

has Lie algebroid $TM \longrightarrow M$.

(3) K Lie group has Lie algebroid $A = \mathrm{Lie}(K)$.

(4) Action groupoid $K \times M \rightrightarrows M$.
This has Lie algebroid the trivial vector bundle $\mathrm{Lie}(K) \times M \longrightarrow M$ w/ anchor $\underbrace{\rho : \mathrm{Lie}(K) \times M \longrightarrow TM}_{\text{the infinitesimal action}}$

(5) $F \subseteq TM$ foliation then $\mathrm{Mon}(F)$ and $\mathrm{Hol}(F)$ have the same Lie algebroid F, ρ is the inclusion to TM.

(6) Let (P, π) be a Poisson manifold $\pi \in \Gamma(\Lambda^2 TP)$, $\underbrace{[\pi, \pi]}_{\text{Schouten bracket}} = 0$

Lie algebroid $A = T^*P$. The anchor map is

$$\rho = i_\pi : T^*P \longrightarrow TP$$

Lie derivative

$$\alpha, \beta \in \Omega^1(P), \quad [\alpha, \beta] = \mathcal{L}_{\rho(\alpha)} \beta - \mathcal{L}_{\rho(\beta)} \alpha - d(\pi(\alpha, \beta))$$

On exact forms, $[df, dg] = d\{f, g\}$.

Crainic-Fernandes: There is no Lie III for Lie algebroids. They also describe the obstructions for integrating Lie algebroids.

Example:

$\mathfrak{g}$ = Lie algebra.

$\mathfrak{g}^*$ is a Poisson manifold w/ the Lie algebroid

$$T^*G \rightrightarrows \mathfrak{g}^*$$

G is a Lie group integrating $\mathfrak{g}$, s, t are the left and right trivializations of T^*G.

symplectic groupoid.

Van Est for Lie groupoids

Classical van Est isomorphism:

G Lie group w/ finite center
$K \leq G$ maximal compact subgroup

van Est: $H^\bullet_{diff}(G) \cong H^\bullet_{Lie}(\mathfrak{g}; K)$

$\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K)$

Lie algebra cochains: $(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k})^*)^K$.

Extensions to Lie groupoids:

$G \rightrightarrows M$ a Lie groupoid. This defines a simplicial manifold:

$G_k = \{(g_1, \dots, g_k) \mid s(g_i) = t(g_{i+1})\}$

face maps:

$$\partial_i : G_K \longrightarrow G_{K-1}$$

$$\partial_i (g_1, \dots, g_K) = \begin{cases} (g_2, \dots, g_K) & i=0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_K) & 1 \leq i \leq k-1 \\ (g_1, \dots, g_{K-1}) & i=k \end{cases}$$

<u>Remark</u>:

$$\partial_i : G_2 \longrightarrow G_1 \quad , \ \partial_0 = s, \ \partial_1 = t$$

<u>Groupoid cohomology</u>

cochains: $C^k_{diff}(G) := C^\infty(G_K)$

differential: $\delta = \sum_{i=0}^{k} (-1)^i \partial_i^* \Longrightarrow \delta^2 = 0$

the cohomology of $\left(\bigoplus_{K \geq 0} C^k_{diff}(G), \delta \right)$ is called the groupoid cohomology, denoted by $H^{\cdot}_{diff}(G)$.

<u>Lie algebroid cohomology</u>

(1) vector bundle $A \longrightarrow M$

(2) Lie bracket on $\Gamma(A)$

(3) anchor map $\rho : \Gamma(A) \longrightarrow TM$.

$$\rho[X,Y] = [\rho(X), \rho(Y)] \qquad [X, fY] = f[X,Y] + (\rho(X)f)Y$$

"de Rham forms" $\Omega_A^k = \Gamma(\Lambda^k A^*)$ w/ differential $d : \Omega_A^k \longrightarrow \Omega_A^{k+1}$

$$(d\omega)(X_1, \dots, X_{K+1}) = \sum_{i<j} (-1)^{i+j-1} \omega([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_{K+1}) +$$

$$\sum_i (-1)^i \rho(X_i) \omega(X_1, \dots, \hat{X}_i, \dots, X_{K+1})$$

Note that $d^2=0$ and hence $\left(\bigoplus_{k\geq 0}\Omega^k_A, d\right)$ is a complex w/ cohomology denoted by $H^\cdot_{Lie}(A)$.

The Weinstein-Xu map.

$$\Phi : H^\bullet_{diff}(G) \longrightarrow H^\cdot_{Lie}(A) \quad , \quad A = \text{Lie algebroid } (G).$$

$$c \in C^k_{diff}(G), \quad X_i \in A$$

$$\Phi(c)(X_1, .., X_k) = \sum_{\sigma\in S_k} (-1)^\sigma R_{X_{\sigma(k)}} \cdots R_{X_{\sigma(1)}} (c)$$

$$(R_X c)(g_2, \dots, g_k) = L_X c(-, g_2, \dots, g_k)(t(g_2))$$

$$\Longrightarrow R_X c \in C^k_{dif}(G)$$

This defines a map $\Phi : C^k_{dif}(G) \longrightarrow \Omega^k_A$ w/ commutes w/ d, δ.

Examples: $M \times M \rightrightarrows M$.

groupoid cohomology: $G_k \cong M^{\times(k+1)} \Longrightarrow C^k_{dif}(G) = C^\infty(M^{\times(k+1)})$

$$(\delta f)(x_0, \dots, x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i f(x_0, .., \hat{x}_i, \dots, x_{k+1}), \quad f \in C^\infty(M^{\times(k+1)}).$$

claim: $H^\cdot_{dif}(M\times M) = \begin{cases} \mathbb{R} & \cdot = 0 \\ 0 & \cdot \neq 0 \end{cases}$

Exercise: prove using the ff contraction:

$$h : C^k_{dif} \longrightarrow C^{k-1}_{dif}$$

$$(hf)(x_0, .., x_{k+1}) = f(x_0, x_0, x_1, .., x_{k+1})$$

$$\Rightarrow \delta h + h\delta = 1.$$

Lie algebroid of $M \times M \rightrightarrows M$ is TM.

$\Rightarrow (\Omega_A^\cdot, d)$ is the usual de Rham complex

$\Rightarrow H^\cdot_{Lie}(TM) \cong H_{dR}(M)$

Remark:

The Weinstein-Xu map factors as follow:

$$C^k_{dif}(M \times M) \longrightarrow C^k_{AS}(M) \longrightarrow \Omega^k_M$$

restriction to the diagonal

Alexander-Spanier cochain complex,

$= germs_{\Delta_{k+1}}(C^\infty(M^{\times(k+1)}))$, i.e

germs of smooth functions around $M \xrightarrow{\Delta_{k+1}} M^{\times(k+1)}$

diagonal

The differential of $C^k_{AS}(M)$ is the "restriction" of δ to the diagonal, denoted by $\hat{\delta}$.

Proposition: $H^\cdot(C^\cdot_{AS}(M), \hat{\delta}) \cong H^\cdot_{dR}(M)$

Theorem (Crainic)

For a proper groupoid $G \rightrightarrows M$,

$(s,t): G \to M \times M$ is proper

$$H^k_{diff}(G) = \begin{cases} C^\infty(M)^{inv} & k=0 \\ 0 & k>0 \end{cases}$$

Ingredients of the proof:

(1) any Lie groupoid has a Haar system i.e a family of measures $\lambda = \{\lambda^x \mid x \in M\}$ on the fibers of the target s.t.

(a) $\forall \phi \in C_c^\infty(G)$, $\int_{t^{-1}(x)} \phi(g)\, d\lambda^x(g) \in C^\infty(M)$

(b) λ is left-invariant

$$\int_{t^{-1}(x)} \phi(gh)\, d\lambda^x(h) = \int_{t^{-1}(y)} \phi(h)\, d\lambda^y(h)$$

(2) On any proper Lie groupoid, there always exists a "cut-off" function $c \in C^\infty(M)$ s.t.

(a) $t : \mathrm{supp}(c \circ s) \longrightarrow M$ proper

(b) $\int_{t^{-1}(x)} c(s(g))\, d\lambda^x(g) = 1 \quad \forall x \in M$

w/ (1) and (2), a contraction of the complex is given by

$$h(\varphi)(g_1, \dots, g_{k-1}) = \int_{t^{-1}(t(g_1))} g' \cdot \varphi((g')^{-1}, g_1, \dots, g_{k-1})\, c(s(g'))\, d\lambda^{t(g_1)}(g').$$

Consider a Lie groupoid action $G \rightrightarrows M \curvearrowright (\mu, Z)$ i.e $\exists\, \mu : Z \to M$ w/ $Z\, {}^{\mu}\!\times_M^{t} G \to Z$. Assume $Z \xrightarrow{\mu} M$ is a surjective submersion. We say that the action is proper if the map

$$\begin{aligned} Z\, {}^{\mu}\!\times_M^{t} G &\longrightarrow Z \times Z \\ (z, g) &\longmapsto (z, zg) \end{aligned}$$

is proper.

Given $G \rightrightarrows M \curvearrowright (\mu, Z)$, there is an associated groupoid $Z\, {}^{\mu}\!\times_M^{t} G \rightrightarrows Z$, the action groupoid.

<u>theorem</u>: Associated to a proper action as above, there is a map $\Phi_Z : H^{\bullet}_{dif}(G) \longrightarrow H^{\bullet}_{F,inv}(Z)$.

μ surjective submersion $\Longrightarrow$ Z is foliated along μ

Call this foliation $F = \ker(d\mu)$. Consider $\Omega^{\bullet}_F := \Gamma(\Lambda^{\bullet} F^*)$. There is a differential d_F (Foliated cohomology).

$G \rightrightarrows M \curvearrowright (\mu, Z) \Longrightarrow G \rightrightarrows M \curvearrowright (\mu|_F, F)$.

Let $\Omega^{\bullet}_{F,inv}$ = invariant foliated forms. This gives $H^{\bullet}_{F,inv}(Z)$.

<u>Back to van Est</u>

H Lie group, K maximal compact subgroup.

$K \backslash H \curvearrowleft H$ properly

w/ $Z = {}_K\backslash^H$ then $F = TZ$.

$\Omega^{k}_{inv}(Z) \cong \Lambda^k(\mathfrak{h}/\mathfrak{k})^K$

Consider Lie group $G \curvearrowright Z$. Consider the ff. double complex

Alexander-Spanier

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & & \\
C^{\infty}_{inv}(Z^2) \arrow[r, hook, "\mathrm{I}"] & C^{\infty}(Z^2) \arrow[u] \arrow[r] & C^{\infty}(Z^2 \times G) \arrow[u] \arrow[r] & C^{\infty}(Z^2 \times G^2) \arrow[u] \arrow[r] & {} & {} \arrow[u, "q"'] \\
 & C^{\infty}(Z) \arrow[u, "d_v"] \arrow[r, "\delta_H"] & C^{\infty}(Z \times G) \arrow[u] \arrow[r] & C^{\infty}(Z \times G^2) \arrow[u] \arrow[r] & {} & {} \arrow[r, "p"'] & {} \\
C^{\infty}_{inv}(Z) \arrow[uu, "d_v"] \arrow[ur, hook] & \mathbb{R} \arrow[u] \arrow[r, "\delta_g"] & C^{\infty}(G) \arrow[u] \arrow[r, "\delta_g"] & C^{\infty}(G^2) \arrow[u] \arrow[r] & {} &
\end{tikzcd}$$

δ_h is the groupoid cohomology differential of the diagonal action $G \curvearrowright Z^k$.
δ_g is the group cohomology differential.

<u>Lemma</u>: The inclusion $\mathbb{R} \longrightarrow C^\infty(Z)$ induces an isomorphism on cohomology.

<u>Proof</u>: Look at filtration by p-degree since cohomology of columns vanishes, except in deg. 1. This proves the lemma.

<u>Lemma</u>: The inclusion $C^\infty_{inv}(Z^2) \longrightarrow C^\infty(Z^2)$ induces an isomorphism on cohomology.

<u>Proof</u>: Look at ~~filtration~~ the rows: this is the groupoid complex for a proper action. This vanishes in cohomology by Crainic's result.

these lemata $\Longrightarrow$ $H^\bullet_{dif}(G) \cong H^\bullet(C^\infty_{inv}(Z^\bullet), d_v)$.

<u>last step</u>: localize the complex $C^\infty_{inv}(Z)$ on the diagonal.

$$\Phi_Z : H^\bullet_{dif}(G) \cong H^\bullet(C^\infty_{inv}(Z^\bullet), \delta_v) \longrightarrow H^\bullet(C^\bullet_{AS, inv}(Z)) \cong H^\bullet_{inv}(Z)$$

ΨDO's on Lie groupoids

Differential operators: Universal Enveloping Algebra

Lie algebra $\mathfrak{g} \longmapsto \mathcal{U}(\mathfrak{g})$ s.t.

$$\begin{tikzcd} & \mathcal{U}(\mathfrak{g}) \arrow[dr, dashed, "\tilde{\phi} \text{ unique}"] & \\ \mathfrak{g} \arrow[ur, hook] \arrow[rr, "\phi"'] & & A \end{tikzcd}$$

Let $A \longrightarrow M$ be a Lie algebroid w/ Lie bracket $[,]$ and anchor map $\rho : A \longrightarrow TM$. The vector space $C^\infty(M) \oplus \Gamma(A)$ has a Lie algebra structure w/

$$[(f, X), (g, Y)] = (\rho(X)g - \rho(Y)f, [X, Y])$$

Consider ~~the~~ its universal enveloping algebra, denoted by $\mathcal{U}$.

Definition

The universal enveloping algebra of A is defined as $\mathcal{U}/_I = \mathcal{U}(A)$ w/ $I = \langle i(f)i(g) - i(fg), i(f)i(X) - i(fX), f, g \in C^\infty(M), X \in \Gamma(A) \rangle$.

Remark

(1) in $\mathcal{U}(A)$, we have $i(f)i(X) = i(fX)$, $[i(X), i(Y)] = i[X, Y]$ and $[i(X), i(f)] = i(\rho(X)f)$.

(2) $\mathcal{U}(A)$ is universal in these relations.

(4) The algebraic version of a Lie algebroid, is called a Lie-Rinehart algebra. The universal enveloping algebra exists for such objects.

Examples

(1) $\mathfrak{g}$ as a trivial Lie algebroid, we get the usual enveloping algebra.

(2) $TM \longrightarrow M$, $\mathcal{U}(TM) = \mathcal{D}(M)$ the algebra of differential operators on M.

(3) $\mathcal{F} \subseteq TM$ foliation, $\mathcal{U}(\mathcal{F})$ is the algebra of differential operators acting on the leaves.

Theorem (Poincaré - Birkhoff - Witt)
The symmetrization map

not needed

$$X_1 \otimes \cdots \otimes X_n \longmapsto \frac{1}{n!}\sum_{\sigma \in S_n} (-1)^{\sigma} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

defines an isomorphism of algebras

$$\Gamma(\mathrm{Sym}\, A) \longrightarrow Gr(\mathcal{U}(A))$$

In particular, $Gr(\mathcal{U}(A))$ is commutative.

Goal: Find a lift $\mathcal{U}(A) \longrightarrow \Gamma(\mathrm{Sym}\, A)$.

Another description of $\mathcal{U}(A)$:

$G \rightrightarrows M \circlearrowright G \xrightarrow{s} M$ and this action lifts to $T_sG := \ker ds \leq TG$.

$G/_G \cong M$, $T_sG/_G \cong A$

$\to \overbrace{C^\infty(M) \cong C^\infty_{inv}(G), \ \Gamma(A)}^{\star} \cong \mathfrak{X}^s_{inv}(G)$ invariant vector fields on the s-fiber

Definition: $\mathcal{D}(G)$ algebra of invariant differential operators along s-fibers.

So $D \in \mathcal{D}(G)$ is actually a smooth family of differential operators $D = \{D_x\}_{x \in M}$, D_x on $s^{-1}(x)$. Invariance means: $D_x = R_g \circ D_y \circ R_{g^{-1}}$ for all $x \xrightarrow{g} y$.

By the isomorphisms $(\star)$ we get maps $\phi_1 : C^\infty(M) \longrightarrow \mathcal{D}(G)$ and $\phi_2 : \Gamma(A) \longrightarrow \mathcal{D}(G)$ satisfying the relations in $\mathcal{U}(A)$. Hence, we get an algebra morphism $\mathcal{U}(A) \xrightarrow{\cong} \mathcal{D}(G)$.

The exponential map on a Lie groupoid.

Choose a connection ∇ on A. Since $t^*A = T_s G$, we get a connection on $T_s G$. This defines the exponential map

$$\exp_{\nabla, x} : A_x \xrightarrow{\cong} (T_s G)_x \xrightarrow{\exp_\nabla} s^{-1}(x) \subseteq G \quad , \ x \in M.$$

Varying the base point, we get $\exp_\nabla : A \longrightarrow G$. It is a local diffeomorphism from a neighborhood of the zero section in A to a neighborhood V of $M \subseteq G$.

Let $\chi \in C^\infty(G)$ s.t.

(a) support$(\chi) \subseteq V$

(b) $\chi \equiv 1$ in a neighborhood of M.

__Definition__: $\xi \in A_x^*$, $\ell_\xi(g) = \chi(g)\, e^{i\langle \exp_\nabla^{-1}(g), \xi\rangle}$ where $\langle , \rangle : A \times A^* \longrightarrow \mathbb{R}$.
$\Longrightarrow \ell_\xi \in C^\infty(G)$.

__Definition__: ~~Thy~~ The symbol $\sigma_\nabla(D) \in C^\infty(A^*)$ of $D \in \mathcal{U}(A)$ is defined by

$$\sigma_\nabla(D)(\xi_x) = (D_x \ell_\xi)(x).$$

__Remark__:
In fact $\sigma_\nabla(D)$ depends polynomially on ξ, for $D \in \mathrm{Filt}_n(\mathcal{U}(A))$ then $\sigma_\nabla(D) \in \bigoplus_{k=0}^{n} \Gamma(\mathrm{Sym}^k A)$.

__Theorem__
The map $D \longmapsto \sigma_\nabla(D)$ defines a linear isomorphism $\mathcal{U}(A) \xrightarrow{\cong} \Gamma(\mathrm{Sym}\, A)$.

__Proof__: σ_∇ respects the filtration so check that its graded quotient is the inverse to the symmetrization map. $\square$

$$\sigma_k : \frac{\mathrm{Filt}_k\, \mathcal{U}(A)}{\mathrm{Filt}_{k-1} \mathcal{U}(A)} \longrightarrow \Gamma(\mathrm{Sym}^k A)$$

We have the ff. property:

$$\sigma_k(D)\sigma_l(E) = \sigma_{k+l}(DE), \quad D \in \mathrm{Filt}_k(\mathcal{U}A),\ E \in \mathrm{Filt}_l(\mathcal{U}A).$$

Consider $[D,E] = 0 \mod \mathrm{Filt}_{k+l-1}(\mathcal{U}(A))$. Define

$$\{\sigma_k(D), \sigma_l(E)\} := \sigma_{k+l-1}[D,E]$$

Lemma: $\{,\}$ defines a Poisson bracket on $\Gamma(\mathrm{Sym}\,A)$.
(Think of $\Gamma(\mathrm{Sym}\,A)$ as polynomials on A^*).

Proof:

Jacobi: $\{\{\sigma_k(D), \sigma_l(E)\}, \sigma_m(F)\} = \sigma_{k+l+m-2}[[D,E],F]$
$\Rightarrow$ Jacobi on $[,]$.

Leibnitz: $\{\sigma_k(D)\sigma_l(E), \sigma_m(F)\} = \sigma_{k+l+m-1}[DE,F]$. $\square$

Explicitly,
$\forall f,g \in C^\infty(M),\ X,Y \in \Gamma(A)$

- $\{f,g\} = 0$
- $\{X,f\} = \rho(X)f$
- $\{X,Y\} = [X,Y]$

Examples:

(1) $\mathfrak{g}$ Lie algebra, $\mathfrak{g}^*$ Lie-Poisson structure.
(2) TM. On T^*M, we have $\{,\}$ as the canonical symplectic structure.

Remark: G acts on T^*_sG w/ quotient $T^*_sG/_G \cong A^*$.

$T^*_s(G) = \bigcup_{x\in M} T^*(s^{-1}(x))$ is a regular Poisson manifold w/ symplectic leaves $T^*(s^{-1}(x))$, $x \in M$.

The quotient map $T^*_s(G) \longrightarrow A^*$ is a Poisson map.

Deformation Quantization

Form the adiabatic Lie algebroid $\{A_\hbar\} \longrightarrow M \times [0,\infty)$.
As a vector bundle, $A_\hbar = A$.

$$\Gamma\{A_\hbar\} = \{[0,+\infty) \xrightarrow{X} \Gamma(A)\}$$

$$[X,Y]_\hbar = \hbar[X_\hbar, Y_\hbar]$$

$$\rho(X)_\hbar = \hbar\rho(X_\hbar)$$

This defines a new algebroid. Consider $\mathcal{U}\{A_\hbar\}$. This algebra contains $C^\infty([0,\infty))$ in its center. Define

$$\varprojlim_n \left(\mathcal{U}(\{A_\hbar\}) \big/ \hbar^n\,\mathcal{U}\{A_\hbar\} \right) \xrightarrow[\cong]{PBW} \Gamma(\mathrm{Sym}\,A)[[\hbar]]$$

In this way, the associated $*$-product on $\Gamma(\mathrm{Sym}\,A)[[\hbar]] \subseteq C^\infty(A^*)[[\hbar]]$. This defines a $*$-product w/ first order term the Poisson bracket $\{,\}$ just defined.

Theorem

$\exists$ an algebra $\Psi^c(G)$ of ΨDO's on G. An element $P \in \Psi^c(G)$ is a family of operators $P = \{P_x\}_{x\in M}$ of ΨDO's on $s^{-1}(x)$ w/:

(a) $x \longmapsto P_x$ is smooth

(b) P is invariant i.e. $P_x = R_g \circ P_y \circ R_{g^{-1}}$, $y \xleftarrow{g} x$.

(c) support condition: $\mu(\mathrm{Supp}(P)) \subseteq G$ compact.

$$\mathrm{Supp}(P) \subseteq \bigcup_{x\in M} (s^{-1}(x) \times s^{-1}(x)) \subseteq G \,{}^s\!\times^s_M G$$

$$\mu : G \,{}^s\!\times^s_M G \longrightarrow G$$
$$(g_1, g_2) \longmapsto g_1 g_2^{-1}$$

Remark:

$$\Psi^{\infty}(G) = \bigcup_m \Psi^m(G)$$

$$\Psi^{-\infty}(G) := \bigcap_m \Psi^m(G) \text{ is an ideal of } \Psi^{\infty}(G).$$

$P \in \Psi^{-\infty}(G)$ has a smooth family of smooth kernels

$$K_{P_x} \in C^{\infty}(s^{-1}(x) \times s^{-1}(x))$$

w/c is invariant under the action

$$K_{P_x}(z_1 g, z_2 g) = K_{P_x}(z_1, z_2).$$

Since $G \overset{s\ s}{\times}_M G / G \xrightarrow[\mu]{\cong} G$ we can consider the reduced kernel $\tilde{K}_P \in C_c^{\infty}(G)$. So $\Psi^{-\infty}(G) \cong C_c^{\infty}(G)$ as algebras. The multiplication in $C_c^{\infty}(G)$ is convolution

$$(f_1 f_2)(g) = \int f_1(gh^{-1}) f_2(h)\, d\lambda(h)$$

A: convolution algebra.

Picture:

$A \subseteq \Psi^{\infty}(G)$

$UA \subseteq \Psi^{\infty}(G)$ as Differential operators

Definition:

$D \in UA$ is elliptic if it is a family of elliptic differential operators on s-fibers.

Ellipticity $\Rightarrow \exists Q \in \Psi^{\infty}(G)$ s.t. $1 - QD,\ 1 - DQ \in A$.

Remark

This is enough to define $\mathrm{ind}(D) \in K_0(A)$. We can pair this w/ $H^{\bullet}_{diff}(G)$. Goal: compute this pairing.

Index Theorems

the case of a proper action of a Lie group.

$G \curvearrowright Z$ properly s.t. Z/G is compact.

D elliptic G-invariant differential operator

Lemma: For a proper, cocompact action $Z \times G \longrightarrow Z$, $\exists$ "cut off" function i.e a $c \in C_c^\infty(Z)$ s.t.

$$\int_G c(xg)\, dg = 1 \quad , \forall x \in Z$$

w/ this function, we can average differential geometric objects to get invariants ones. For example, let ρ be a Riemannian metric. then

$$\rho^{ave}_x (X, Y) = \int_G c(xg) \rho (Xg, Yg)\, dg$$

There exists an algebra of G-invariant ΨDO's on Z,

$$\Psi^\infty_{inv}(Z) = \bigcup_m \Psi^m_{inv}(Z) \ni P$$

(i) $P = R_g \circ P \circ R_{g^{-1}}$

(ii) $supp(P) \subseteq Z \times Z$ is G-compact

* look at the paper of Connes & Moscovici about L^2-index

Choose a G-invariant Riemannian metric, so we get

$$exp : TZ \longrightarrow Z \times Z$$

define $e_\xi(x, y) = \chi(x,y) e^{i\langle \xi, \exp_x^{-1}(y)\rangle}$

$$\chi(x,y) \equiv \begin{cases} 1 & \text{in a neighborhood of } \Delta \\ 0 & \text{outside another one.} \end{cases}$$

Let $a \in C^\infty(T^*Z)$,

$$[Op(a)(f)](x) = \int_{T_x^*Z} \int_Z e_\xi(x,y)\, a(x,\xi) f(y)\, dy\, d\xi$$

For a suitably chosen $a \in C^\infty(T^*Z)$, this defines a G-invariant ΨDO. The kernel of this operator is

$$K_{Op(a)}(x,y) = \int_{T_x^*M} e_\xi(x,y)\, a(x,\xi)\, d\xi$$

$$Op(a) \in \Psi^\infty_{inv}(Z) \Longrightarrow K_{Op(a)}(xg, yg) = K_{Op(a)}(x,y).$$

Smoothing operators : $\Psi^{-\infty}_{inv}(Z) \subseteq C^\infty_{G\text{-comp}}(Z \times Z)^G$

<u>The Trace</u> :

Define the ff. functional $\tau : \Psi^{-\infty}_{inv}(Z) \longrightarrow \mathbb{C}$,

$$\tau(K) := \int_Z K(x,x)\, c(x)\, dx$$

$\nwarrow$ volume of the metric

<u>Lemma</u> : when G is unimodular, τ is a trace.

<u>Proof</u>:

$$\tau[K_1, K_2] = \int_Z c(x_1)\big(K_1(x_1,x_2) K_2(x_2,x_1) - K_2(x_1,x_2) K_1(x_2,x_1)\big)\, dx_1\, dx_2$$

$$= \int_G \varphi(g)\, dg$$

$$\varphi(g) = \iint_{Z\,Z} c(x_1)\, c(x_2 g)\, [K_1, K_2]\, dx_1\, dx_2$$

Now,

$$\varphi(g^{-1}) = \iint_{Z\,Z} c(x_1)c(x_2)\big(K_1(x_1, x_2 g)K_2(x_2 g, x_1) - K_2(x_1, x_2 g)K_1(x_2 g, x_1)\big)\,dx_1\,dx_2$$

$$= \iint_{Z\,Z} c(x_1)c(x_2)\big(K_1(x_1 g^{-1}, x_2)K_2(x_2, x_1 g^{-1}) - \ldots$$

$$= \int_Z\int_Z c(x_1 g)\,c(x_2)\,\big(K_1(\ldots$$

$$= -\varphi(g) \qquad \text{unimodularity} \Rightarrow dg = dg^{-1}$$

$$\Rightarrow \tau[K_1, K_2] = 0 \Rightarrow \tau \text{ is a trace. } \square$$

<u>The Index Pairing</u>

D elliptic, $\exists$ parametrix P s.t. $\underbrace{1-PD}_{S_0}, \underbrace{1-DP}_{S_1} \in \Psi^{-\infty}_{inv}(Z)$.

Consider the ff. operator matrix:

$$L = \begin{pmatrix} S_0 & -(P+S_0P) \\ D & S_1 \end{pmatrix} \in M_2(\Psi^{-\infty}_{inv}(Z))$$

$$L^{-1} = \begin{pmatrix} S_0 & (1+S_0)P \\ -D & S_1 \end{pmatrix}$$

Let $R_D := \underbrace{L\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}L^{-1}}_{\pi^2=\pi} - \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{Q^2=Q} = \begin{pmatrix} S_0^2 & S_0(1+S_0)P \\ S_1D & -S_1^2 \end{pmatrix}$

$\cap$

$M_2(\Psi^{-\infty}_{inv}(Z))$

Remark:

Suppose G is the trivial group, this means that Z is compact. and $C \equiv 1$. Then $\tau(R_D) = \mathrm{Trace}(S_0^2) - \mathrm{Trace}(S_1^2) = \mathrm{index}(D)$.

Higher Index

Recall that any $\alpha \in H^k_{diff}(G)$ can be represented by a function $\varphi_\alpha \in C^\infty_{inv}(Z^{k+1})$.

$$\begin{tikzcd}
C^\infty_{in}(Z^2) \arrow[r] & C^\infty(Z^2 \times G) \arrow[r] & {} \\
C^\infty_{inv}(Z) \arrow[r] \arrow[u, "\delta"] & C^\infty(Z^2 \times G) \arrow[r] \arrow[u] & {} \\
 & C^\infty(G) \arrow[r, "\delta"'] \arrow[u] & C^\infty(G^2) \arrow[u]
\end{tikzcd}$$

$(\delta\varphi)(z_0, .., z_k) = \sum_i (-1)^i \varphi(z_0, \dots \hat{z}_i, \dots z_k)$

acyclic

Definition:

Let $K_0, \dots, K_n \in \Psi^{-\infty}_{in}(Z)$ and $\varphi \in C^\infty_{inv}(Z^{n+1})$

$$\langle \varphi, K_0 \otimes \cdots \otimes K_n \rangle = \int_{Z^{n+1}} C(x_0)\varphi(x_0, .., x_n) K_0(x_0, x_1) K_1(x_1, x_2) \dots K(x_n, x_0)\, dx_1 \dots dx_n$$

Lemma:

$$\langle \varphi, b(K_0 \otimes \cdots \otimes K_n) \rangle = \langle \delta\varphi, K_0 \otimes \cdots \otimes K_n \rangle$$

Define $\mathrm{Ind}_\alpha(D) := \langle \varphi_\alpha, ch(R_D) \rangle$.

With the lemma above, this construction defines a map

$$H^\cdot_{diff}(G) \longrightarrow HC^\cdot(\Psi^{-\infty}_{inv}(Z))$$

For the trivial class, $[1] \in H^0(G)$, one gets the trace. τ.

Remark on G-invariant cohomology

$\Omega^{\bullet}_{inv}(Z) \subseteq \Omega^{\bullet}(Z)$, $\Omega^{\bullet}_{inv}(Z) = \{\alpha \in \Omega^{\bullet}(Z) \mid g^*\alpha = \alpha, \forall g \in G\}$

d_{dR} restricts to $\Omega^{\bullet}_{inv}(Z)$.

$H^{\bullet}_{inv}(Z) = H^{\bullet}_{inv}(\Omega^{\bullet}_{inv}(Z), d_{dR})$.

Lemma: The integral

$$\int_Z c\alpha \quad w/\ \alpha \in \Omega^{top}(Z)$$

vanishes on exact forms.

Proof:

$$\int_Z c\, d\beta = -\int_Z dc \wedge \beta = \int_Z\int_G (g^{-1})^* c\, \beta \wedge dc = \int_Z\int_G c\beta \wedge g^* dc = 0$$

this follows from differentiation of the identity $1 = \int_G c(xg^{-1})$. $\square$

Asymptotic ΨDO's

$Op_\hbar(a) = Op(a(x, \hbar\xi))$

We get a product on symbols by setting

$$a \circ_\hbar b = \begin{cases} \sigma_\hbar(Op_\hbar(a)\, Op_\hbar(b)) & \hbar > 0 \\ a \cdot b & \hbar = 0 \end{cases}$$

$\sigma_\hbar = i_{\hbar^{-1}} \cdot \sigma$, $i_\hbar(a(x,\xi)) = a(x, \hbar\xi)$

defines an associative product whose jet expansion at $\hbar = 0$ defines a $*$-product on T^*Z.

The package

$$Op_\hbar(a)\, Op_\hbar(b) \sim Op_\hbar(a*b) \quad \text{as } \hbar \to 0$$

$$\text{trace } \tau(K) = \int c(x) K(x,x)\, dx, \quad K_{Op(a)}(x,y) = \int_{T_x^*M} e_\xi(x,y)\, a(x,\hbar\xi)\, d\xi$$

$$\Longrightarrow \tau(K_{Op_\hbar(a)}) = \int_{T^*M} c(x)\, a(x,\xi)\, dx\, d\xi$$

induces on $*$-deformed algebra a trace.

Compute $\quad \alpha \in H^{2k}_{diff}(G)$

$$Ind_\alpha(D) = \langle \varphi_\alpha, R_D \otimes \cdots \otimes R_D \rangle$$

$$= \tau(\varphi_0 R_D \varphi_1 \cdots \varphi_{2k} R_D) \qquad \varphi = \varphi_0 \otimes \cdots \otimes \varphi_n$$

$$= \tau(\varphi_0\, Op(a)\, \varphi_1 \cdots \varphi_{2k}\, Op(a))$$

$[R_D = Op(a)]$

$$= \lim_{\hbar \to 0} \tau(\varphi_0\, Op_\hbar(a)\, \varphi_1 \cdots \varphi_{2k}\, Op_\hbar(a))$$

$$= \lim_{\hbar \to 0} \tau(Op_\hbar(\varphi_0 * a) \cdots Op_\hbar(\varphi_{2k} * a))$$

$$= \lim_{\hbar \to 0} \tau(Op_\hbar(\varphi_0 * a * \varphi_1 * \cdots * \varphi_{2k} * a))$$

$$= \tau_\hbar(\varphi_0 * \cdots * a).$$

Example

① $G = \{1\} \quad a * a = a$

Fedosov; Nest-Tsygan Index theorem : $Ind(D) = \int_{T^*Z} Td(T^*Z)\, ch(a_0)$. $\sigma(D)$

② $G \neq \{1\},\ \alpha = [1]$.

The $*$-product induced on T^*Z is G equivariant.

$(A_{T^*Z})^G; \quad HC_\cdot^{per}(A^G_M) \xrightarrow[\text{reduction}]{\cong} H^\cdot_{inv}(Z)$

This gives us $\langle \tau, R_D \rangle = \int_{T^*Z} c\, Td(T^*Z)\, ch(\sigma(D))$

Theorem (General case)

$$Ind_\varphi(D) = \int_{T^*Z} \underbrace{\varphi_0\, d\varphi_1 \wedge \dots \wedge d\varphi_{2k}}_{\Phi_Z(\alpha)} \wedge Td(T^*Z)\, ch(\sigma(D))$$

van Est map

Theorem (Pflaum-Posthuma-Tang)

$$Ind_\alpha(D) = \int_{T^*Z} \Phi_Z(\alpha)\, Td(T^*Z)\, ch(\sigma(D))$$

Remark:
When G is not unimodular, there is no trace but there are higher cyclic cocycles:

$$H^\cdot_{diff}(G; \Lambda^{top}\mathfrak{g}) \longrightarrow HC^\cdot(\Psi^{-\infty}_{inv}(Z))$$

Theorem: Let $G \rightrightarrows M \curvearrowright Z \xrightarrow{\mu} M$ properly, μ surjective submersion. Assume G to be unimodular. Then $\forall \alpha \in H^{2k}_{diff}(G)$ and $D = \{D_x\}_{x\in M}$ a smooth G-invariant family of elliptic diff. operators D_x on $\mu^{-1}(x)$, there is a similar definition of the higher index $Ind_\alpha(D)$ and

$$Ind_\alpha(D) = \int_{T^*_\mu Z} c\, \Phi_Z(\alpha)\, Td(T^*_\mu Z)\, ch(\sigma(D)).$$

Remark
The proof is the same. The relevant Poisson manifold is $T^*_\mu Z$ a regular Poisson so Fedosov machinery applies.

Corollary: $G \rightrightarrows M \curvearrowright G \xrightarrow{s} M$, $D \in \mathcal{U}(A)$ be elliptic then

$$Ind_\alpha(D) = \int_{A^*} \Phi_G(\alpha)\, Td(A^*)\, ch(\sigma(D)).$$

Weinstein-Xu map.

Example:

(1) Pair groupoid. $M \times M \rightrightarrows M$, M compact.

proper, so $H^\bullet_{diff}(M \times M) = 0$, $\bullet > 0$.

The trivial cocycle gives Atiyah-Singer index.

(2) Connes-Moscovici (91).

$$\begin{tikzcd} \tilde{M} \arrow[r] & M \end{tikzcd}$$

↑ universal cover

$$\Gamma = \pi_1(M, x_0)$$

$\tilde{M} \times_\Gamma \tilde{M} \rightrightarrows \tilde{M}$ a Lie group.

Its Lie algebroid is $TM \to M$.

Remark:

(1) $\tilde{M} \times_\Gamma \tilde{M} \sim_{\text{morita}} \Gamma$

(2) groupoid cohomology is invariant under morita [crainic].

(3) $H^\bullet_{diff}(\tilde{M} \times_\Gamma \tilde{M}) \cong H^*_{gp}(\Gamma) \cong H^\bullet(B\Gamma)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underset{\alpha}{\Psi}$

$$\mathrm{Ind}_\alpha(D) = \int_{T^*M} \phi_G(\alpha)\, Td\, ch(\sigma(D))$$

$$\Phi_G : H^\bullet(B\Gamma) \xrightarrow{\psi^*} H^\bullet(M)$$

$$\begin{tikzcd} \tilde{M} \arrow[d] & \\ M \arrow[r, "\psi"'] & B\Gamma \end{tikzcd}$$