Deformation Quantization and Operators

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$$f = \sum_{k=0}^{n} P_k(x)\xi^k \longmapsto Op(f) : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$$

$\forall k,\ P_k(x) \in C^\infty(\mathbb{R})$

$$Op(f) = \sum_{k=0}^{n} P_k(x)\left(\frac{\partial}{\partial x}\right)^k$$

Improvements

(a) Introduce parameter $\hbar > 0$

$$Op_\hbar(f) = \sum_{k=0}^{n} P_k(x)\left(i\hbar\frac{\partial}{\partial x}\right)^k$$

$$[Op_\hbar(\xi), Op_\hbar(x)] = i\hbar\,\mathbb{1}$$

In general,

$$Op_\hbar(f) \cdot Op_\hbar(g) = Op_\hbar(f * g)$$

$$f * g = fg + \sum_{k=1}^{\infty} \frac{(i\hbar)^k}{k!}\partial_\xi^k f\, \partial_x^k g \qquad (\#)$$

$$f*g - g*f = i\hbar(\partial_\xi f\,\partial_x g - \partial_x f\,\partial_\xi g) + \ldots = i\hbar\{f,g\} + O(\hbar^2)$$

Remark

(1) If f and g are polynomials then the sum (#) is finite.

(2) In $\{f,g\}$, x and ξ are on the same footing while in $*$ they are not.

Why not $Op_\hbar(f) = \sum_{k=0}^{n} (i\hbar\partial_x)^k P_k(x)$? The SYMMETRIC choice (H. Weyl).

$f, g \in \mathbb{C}[x,\xi]$, $Op_\hbar(x^p\xi^q) = \frac{1}{(p+q)!}\sum Y_{s_1} Y_{s_2} \cdots Y_{s_{p+q}}$

$$Y_1 = Y_2 = \dots = Y_p = x$$
$$Y_{p+1} = \dots = Y_{p+q} = i\hbar\frac{\partial}{\partial x}$$

$$Op_\hbar(f) \circ Op_\hbar(g) = Op_\hbar(f \underset{\omega}{*} g)$$

$$f \underset{\omega}{*} g = \sum_{k=0}^{n} \frac{(i\hbar)^k}{k!} A_k(f,g) \quad , \quad A_k(f,g) = \sum_{\alpha,\beta} a_{\alpha,\beta}\, \partial^\alpha f\, \partial^\beta g$$

$$f *_\omega g = \left[\exp\left(\frac{i\hbar}{2}(\partial_\xi \partial_y - \partial_\eta \partial_x)\right) f(x,\xi) g(y,\eta) \right]\Bigg|_{x=y,\ \xi=\eta}$$

On $\mathbb{C}[x,\xi]$, the group of linear symplectomorphisms
i.e $\mathbb{R}^{2n} \xrightarrow{g} \mathbb{R}^{2n}$ linear ; $g^*(d\xi \wedge dx) = d\xi \wedge dx$
$*_\omega$ is invariant under g, $\{f,g\}$ as well.

Let X be a manifold.

$$f * g = fg + \sum_{k=1}^{\infty} (i\hbar)^k A_k(f,g)$$

linear bidifferential expressions
i.e
$$C^\infty(X) \otimes C^\infty(X) \longrightarrow C^\infty(X)$$
in local coordinates they are
$$\sum_{|\alpha|,|\beta| \leq N} A_{\alpha,\beta}(x)\, \partial^\alpha f\, \partial^\beta g$$

The product $*$ satisfies:

(1) $1 * f = f * 1 = f$

(2) $f*(g*h) = (f*g)*h$ in $C^\infty(X)[[\hbar]]$

Isomorphisms of deformation quantizations:

$* \sim *' \iff \exists T : C^\infty(X)[[\hbar]] \longrightarrow C^\infty(X)[[\hbar]]$, $T = id + \sum_{k=1}^{\infty} (i\hbar)^k T_k$,

T_k linear differential operators s.t.

$$T(f*g) = T(f) *' T(g)$$

Given such $*$: $\{f,g\} = \frac{1}{i\hbar}(f*g - g*f)\Big|_{\hbar=0} = A_1(f,g) - A_1(g,f)$

$\Longrightarrow$ (1) $\{f, gh\} = \{f,g\}h + g\{f,h\}$

(2) $\{,\}$ is a Lie bracket

(2) $* \sim *'$ then $\{,\} = \{,\}'$.

How do we procede from here?

(1) classify all $*$ on a given X up to isomorphisms.

(a) $\{,\}$ corresponds to a symplectic structure on X (Fedosov).

(b) In general, $\{,\}$ is much harder (Kontsevich '97)

(2) Application to index theorems.

(3) Look at modules $(C^\infty(X)[[\hbar]], *)$.

(4) Something else later.

Index Theory

$$D : H_+ \longrightarrow H_- \quad \text{Fredholm operator}$$

$$\operatorname{ind}(D) = \dim \ker(D) - \dim \operatorname{coker}(D)$$

How to compute ind(D)?

Assume we have:

$$h > 0,\ f \in C^\infty(M) \text{ for some } M \rightsquigarrow Op_\hbar(f) : \mathcal{H}_+ \longrightarrow \mathcal{H}_-$$

OR

$$Op_\hbar(f) : \mathcal{H} \longrightarrow \mathcal{H}$$

$$Op_\hbar(f) \circ Op_\hbar(g) = \sum_{k=0}^{N} (i\hbar)^k \, Op_\hbar(A_k(f,g)) + O(h^{N+1}) \quad \forall N$$

Suppose $D = Op_\hbar(F)\big|_{\hbar=1}$ and suppose $Op_\hbar(F)$ is Fredholm $\forall \hbar > 0$.
Then $\operatorname{ind}(Op_\hbar(F))$ stays the same.

$$\operatorname{ind}(D) = \operatorname{Trace}\left(e^{-D^*D} - e^{-DD^*}\right)$$

$$\operatorname{ind}(D) = \operatorname{Trace}\left(e^{-D_\hbar^* D_\hbar} - e^{-D_\hbar D_\hbar^*}\right) = \lim_{\hbar \to 0} \operatorname{Tr}\left(e^{-D_\hbar^* D_\hbar} - e^{-D_\hbar D_\hbar^*}\right)$$

Assume:

$$\operatorname{Trace} Op_\hbar(f) = \frac{1}{(i\hbar)^p}\left(T_0(f) + i\hbar T_1(f) + \dots\right)$$

$f \in (\text{subspace}) \subseteq C^\infty(M)$ $\qquad$ $T_k(f) = (\text{some distribution})_k(f)$

BTW: $f \in \mathbb{C}[x, \xi]$

$Op_\hbar(f)$ is NOT TRACE CLASS.
This is fine since D is neither.

Assume: $\underset{A(M)}{\underset{\shortparallel}{}}\ (\underbrace{C^\infty(M)[[\hbar]], *}_{\text{sheaf of algebras}})$ deformation quantization

$$\tau : C_c^\infty(M)[[\hbar]] \longrightarrow \mathbb{C}((\hbar)) \qquad (\text{or extended to } S(M))$$

$$\tau(a * b - b * a) = 0$$

$$\tau(f) = \frac{1}{(i\hbar)^p} \sum_{k=0}^{\infty} (i\hbar)^k \tau_k(f)$$

$D \in M_N(C^\infty(M)[[\hbar]])$, we want to compute $\tau(e^{-D^*D} - e^{-DD^*}) \in \mathbb{C}((\hbar))$.

$a \in M_N(A(M))$, $e^a = \sum_{k=0}^{\infty} \frac{a^{*k}}{k!}$ (make sense).

<u>The Projection</u>

$\underset{\text{operator}}{A} : \mathcal{H}_+ \longrightarrow \mathcal{H}_- \quad \rightsquigarrow \quad P_A : \mathcal{H}_+ \oplus \mathcal{H}_- \longrightarrow \mathcal{H}_+ \oplus \mathcal{H}_-$

orthogonal projection onto the graph of A.

Explicitly,

$$P_A = \begin{pmatrix} 1 + A^*A & (1 + A^*A)^{-1}A^* \\ A(1 + A^*A)^{-1} & A(1 + A^*A)^{-1}A^* \end{pmatrix}$$

$$\Rightarrow \operatorname{Tr}\Big(P_A - \underset{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}{\cancel{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}}\Big) = \operatorname{Tr}\big((1 + A^*A)^{-1}\big) - \operatorname{Tr}\big((1 + AA^*)^{-1}\big)$$

projection onto the graph of A

projection onto $\mathcal{H}_-$

Actually, given D

$$e^{-D^*D} = (1+A^*A)^{-1}$$

$$e^{-DD^*} = (1+AA^*)^{-1} \qquad A = Df(D^*D)$$

$$f(x) = \frac{e^x - 1}{x}$$

or rather, $\tau: C_c^\infty(M)[[\hbar]] \longrightarrow \mathbb{C}((\hbar))$ on $A(M)$

$$\tau(f) = \frac{1}{\hbar^p} \sum_{k=0}^{\infty} (i\hbar)^k \tau_k(f)$$

$$P^2 = P, \; Q^2 = Q \in M_N(\mathbb{A}(M))$$

$$P - Q \in M_N(\mathcal{S}(M))[[\hbar]]$$

(suppose $*$, τ respects $\mathcal{S}(M)$ i.e $\tau_k \in \mathcal{S}(M)$.)

$$\underbrace{\tau(P-Q) = ?}_{\text{concerns of algebraic index theorems.}}$$

<u>Examples</u>

$a = f(x, \xi) \longrightarrow \infty$ as $\|x|^2 + |\xi|^2 \longrightarrow \infty$.

$$e^{-a^*a} = \sum_{k=0}^{\infty} \frac{(-a * a)^{*k}}{k!} = e^{-\bar{a}a} \underbrace{(1 + \hbar + \hbar^2 + \ldots)}_{\text{each coeffients } \in \mathcal{S}(\mathbb{R}^2).}$$

More on $f \longmapsto Op_\hbar(f)$: $h(x) \longmapsto \int_{\mathbb{R}} K_f(x,y) h(y)\, dy$

$$f = \xi, \quad h(x) \longmapsto i\hbar \frac{\partial h(x)}{\partial x} = i\hbar \int_{\mathbb{R}} \delta'(x-y) h(y)\, dy$$

supported on $y = x$.

$c\,\delta'(z) = \underset{\xi \to z}{\text{Fourier}}\; \xi$

$$i\hbar\,\delta'(x-y) = \underset{\xi\to x-y}{\mathrm{Fourier}^{\hbar}}(\xi)$$

$$\underset{x-y}{\mathrm{Fourier}^{\hbar}} F = \frac{1}{(2\pi\hbar)^{1/2}} \int_{\mathbb{R}} e^{\frac{i(x-y)\xi}{\hbar}} F(\xi)\,d\xi$$

The operator has kernel $K_f(x,y) = \frac{1}{(2\pi\hbar)^{n/2}} \int_{\mathbb{R}^n} e^{\frac{i}{\hbar}(x-y)\xi} f\left(\frac{x+y}{2},\xi\right) d\xi$

$f \in C_c^\infty(\mathbb{R}^{2n}) \Longrightarrow K_f(x,y)$ smooth & properly supported

What about other operators? say w/ kernel $K(x,y) = \int e^{\frac{\varphi(x,y,\theta)}{i\hbar}} a(x,y,\theta)\,d\theta$

<u>Continuation</u>

$A(\mathbb{R}^{2n}) = C^\infty(\mathbb{R}^{2n})[[\hbar]]$

$\xi, x = (\xi_1, \dots, \xi_n, x_1, \dots, x_n)$

$$(f * g)(x,\xi) = e^{\frac{i\hbar}{2}(\partial_y\partial_\xi - \partial_x\partial_\eta)} f(x,\xi)\,g(y,\eta)\Big|_{x=y,\ \xi=\eta}$$

$\uparrow$ the Weyl product

<u>Remark</u>:

Let $Sp(2n,\mathbb{R}) = \{ \mathbb{R}^{2n} \underset{g\ \text{linear}}{\xrightarrow{\sim}} \mathbb{R}^{2n} \mid g^*(d\xi \wedge dx) = d\xi \wedge dx \}$

$Sp(2n,\mathbb{R}) \curvearrowright A_M$ preserving $*$.

$$\omega = d\xi \wedge dx = \sum_{k=1}^{n} d\xi_k \wedge dx_k$$

$$[\xi_k, x_j] = i\hbar\,\delta_{jk}, \quad [\xi_i, \xi_k] = [x_i, x_k] = 0$$

$A(\mathbb{R}^2)$-modules

Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$

Consider 1-parameter families of functions / formal expressions:

$$\mathbb{V}_\varphi = \{ e^{\frac{\varphi(x)}{i\hbar}} a(x) \mid a \text{ smooth} \}[\![\hbar]\!] \qquad \hbar > 0 \text{ or formal}$$

$$\xi (e^{\varphi/i\hbar} a) = e^{\varphi/i\hbar} (i\hbar \partial_x a + \varphi' a)$$

In other words, at $\varphi = 0$ it extends to

$$f(x,\xi)\, a(x) = \sum_{k=0}^{\infty} \frac{(i\hbar)^k}{k!} \underbrace{(\partial_\xi^k f)(\partial_x^k a(x))}_{\text{modified (?)}} \Big|_{\xi = 0}$$

__Remarks__:

(1) $f \cdot a$ depends only on $f(x,\xi)|_{\xi=0}$

(2) $f \cdot_\varphi a$ depends only on $f|_{\{\xi = \varphi'(x)\}}$

(3) x acts by multiplication in all cases.

(4) In general, $f(x,\xi) \cdot_\varphi a(x) = f(x, \xi + \varphi'(x)) \cdot_{\varphi = 0} a(x)$

Now, everything is symmetric so there should be a module supported at $x = 0$. One way is using the Fourier automorphism

$$(Ff)(x,\xi) = f(-\xi, x)$$

$$f \cdot a(\xi) = Ff \cdot a$$

__Differential Operators on Distributions__

start acting by $f(x, i\hbar \frac{\partial}{\partial x})$

$$\begin{tikzcd} \delta_0 \arrow[r, "f(x, i\hbar\frac{\partial}{\partial x})"'] & i\hbar\,\delta_0' \arrow[r] \arrow[l, bend left, "\frac{1}{i\hbar} x"] & (i\hbar)^2 \delta_0'' \arrow[r] \arrow[l, bend left] & \cdots \arrow[l, bend left] \end{tikzcd}$$

So we get linear combinations of $\frac{(i\hbar)^k \delta_0^{(k)}}{k!} =: \xi^k$

Generalization, choose $\psi(\xi)$ smooth and consider $\hbar$-dependent distributions $u(x,\hbar) = \frac{1}{\sqrt{2\pi\hbar}} \int \exp\left(\frac{x\xi + \psi(\xi)}{i\hbar}\right) b(\xi)\, d\xi$, b smooth.

<u>claim</u>: there is a well-defined classical way to make sense of it as a distribution.

<u>Example</u>: $\psi = 0$, $b = \xi^k \Rightarrow u \sim \delta_0^{(k)}$.

So consider those $u(x,\hbar)$. How does $A(\mathbb{R}^{2n})$ act?

$$x\, u(x,\hbar) = \frac{1}{\sqrt{2\pi\hbar}} \int x \exp\left(\frac{x\xi + \psi(\xi)}{i\hbar}\right) b(\xi)\, d\xi$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int \left(i\hbar \frac{\partial}{\partial\xi} - \psi'(\xi)\right) \exp\left(\frac{x\xi + \psi(\xi)}{i\hbar}\right) b(\xi)\, d\xi$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int \exp\left(\frac{x\xi + \psi(\xi)}{i\hbar}\right) \left(-\psi'(\xi) - i\hbar \frac{\partial}{\partial\xi}\right) b(\xi)\, d\xi$$

So,

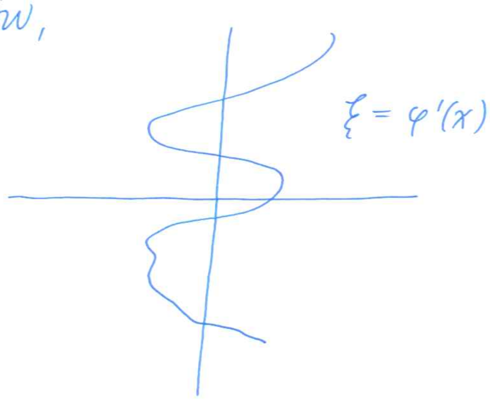
$$x : b(\xi) \longmapsto -(\psi'(\xi) b(\xi) + b'(\xi))$$
$$\xi : b(\xi) \longmapsto \xi b(\xi)$$

since $\xi \cdot u(x,\hbar) = \frac{1}{\sqrt{2\pi\hbar}} \int \frac{\partial}{\partial x} \exp\left(\frac{x\xi + \psi(\xi)}{i\hbar}\right) b(\xi)\, d\xi$

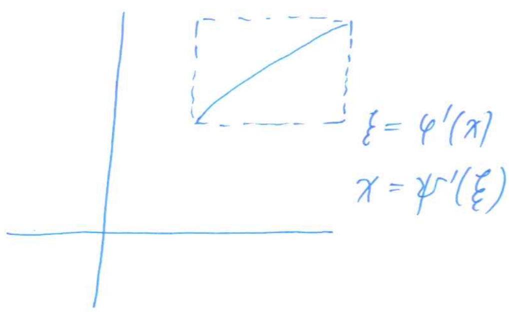
$$= \frac{1}{\sqrt{2\pi\hbar}} \int \exp\left(\frac{x\xi + \psi(\xi)}{i\hbar}\right) \xi\, b(\xi)\, d\xi$$

This is $\mathbb{V}_{\pm\psi}$ conjugated by the automorphism F. It is supported on $x = \psi'(\xi)$ i.e $f \cdot a$ depends on $f(x,\xi)|_{x=\psi'(\xi)}$.

Now,



locally, can be viewed as

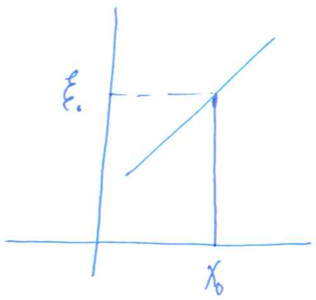


$$u(x) = \exp\left(\frac{\varphi(x)}{i\hbar}\right) a(x)$$

$$v(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \exp\left(\frac{x\xi + \psi(\xi)}{i\hbar}\right) b(\xi)\, d\xi$$

One can ask if and in what sense these are the same family of generalized functions.

<u>Example</u>: linear



$$\xi = k(x - x_0) + \xi_0 = \varphi'(x)$$
$$x = k^{-1}(\xi - \xi_0) + x_0 = \psi'(\xi)$$

$\Longrightarrow$ up to a constant,

$$\varphi(x) = \frac{k}{2}(x - x_0)^2 + \xi_0 (x - x_0)$$
$$\psi(x) = \frac{k^{-1}}{2}(\xi - \xi_0)^2 + x_0(\xi - \xi_0)$$

So given $b(\xi)$, say $b(\xi) = 1$,

$$v(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \exp\left(\frac{x\xi}{i\hbar}\right) \exp\left(\frac{k^{-1}}{2i\hbar}(\xi - \xi_0)^2\right) \exp\left(\frac{x_0(\xi - \xi_0)}{i\hbar}\right) d\xi$$

$$= \text{Fourier}_\hbar \left(\exp\left(\frac{k^{-1}}{2i\hbar}(\xi - \xi_0)^2 + \frac{1}{i\hbar}(\xi - \xi_0) x_0 \right) \right)$$

$$= \frac{1}{\sqrt{A}} \exp\left(-\frac{A^{-1}x^2}{2}\right)\Bigg|_{x=0} \qquad \text{where } A = \frac{-\hbar^{-1}}{i}$$

$$v(x) = \frac{1}{\sqrt{2\pi\hbar}} \int \exp\left(\frac{x\xi + \psi(\xi)}{i\hbar}\right) b(\xi)\, d\xi$$

$$u(x) = \exp\left(\frac{\varphi(x)}{i\hbar}\right) a(x)$$

depends on φ, ψ

$$b(\xi) = 1 \Longrightarrow a(x) = \frac{|k|^{1/2}}{\sqrt{\pm i}} \exp(c/i\hbar)$$

depends on sign of k

What if $b(\xi)=\xi$?

$$v(x) = \text{Fourier}_\hbar\left(\xi \exp\left(-\frac{x}{i\hbar}\right)\right) = \pm i\hbar \frac{\partial}{\partial x} \text{Fourier}_\hbar\left(\exp\left(-\frac{x}{i\hbar}\right)\right)$$

$$= (kx + \hbar \dots)\, \text{Fourier}_\hbar\left(\exp\left(-\frac{x}{i\hbar}\right)\right)$$

Similarly, for $b(\xi)=\xi^p$,

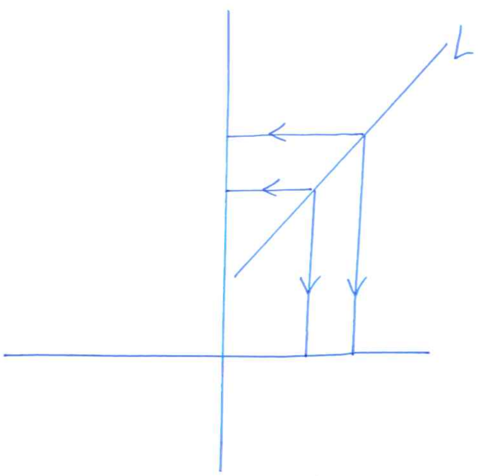
$$v(x) = \pm i\hbar \frac{\partial^p}{\partial x^p} \text{Fourier}_\hbar\left(\exp -\frac{x}{i\hbar}\right)$$

Conclusion:

$$a(x) = \exp(c/i\hbar)\, \frac{|k|^{1/2}\, b(kx) + \hbar ..}{\sqrt{\mp i}}$$

depends on the sign of k

Reminder
When we deal w/ $\sqrt{\ }$ we take the branch cut at the negative real axis.



$a(x) = b(kx) \mod \hbar$

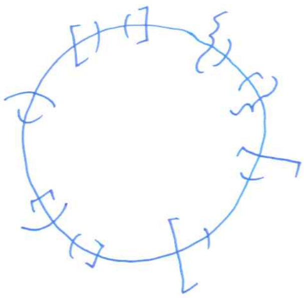
they define the same function on L

$a(x)\,|dx|^{1/2} = b(k\xi)\,|dx|^{1/2}$

as half-densities on L

Let $L \subseteq \mathbb{R}^2$ be a curve, maybe closed. Consider open charts $\{U_i\}$ s.t. $x = \psi'_\delta(\xi)$ on U_δ, $\xi = \varphi'_\delta(x)$

On intersections, we want this to both work.



$$\mathbb{K} \otimes_{\mathbb{C}[[\hbar]]} \Gamma(L, |\Omega_L|^{1/2})[[\hbar]] = W_L$$

$$\mathbb{K} = \left\{ \sum_{p=0}^{N} \cancel{\alpha_p} \exp\left(\frac{a_p}{i\hbar}\right) \,\middle|\, a_p \in \mathbb{R} \right\}$$

Novikov ring

Consider $g_{\alpha\beta} : W_L \big|_{U_\alpha \cap U_\beta} \longrightarrow W_L \big|_{U_\alpha \cap U_\beta}$

$$g_{\alpha\beta} = \varepsilon_{\alpha\beta} \exp\left(\frac{c_{\alpha\beta}}{i\hbar}\right)$$

$c_{\alpha\beta} \in H^1(L, \mathbb{R})$

↑ one-form ξdx

$\varepsilon_{\alpha\beta} \in \check{H}^1(L, \{e^{k\frac{i\pi}{4}}\}_{k\in\mathbb{Z}})$

↑ Maslov class

$\mathbb{V}_L = W_L$ glued by $\{g_{\alpha\beta}\}$

↑ $A(\mathbb{R}^2)$-module structure

So corresponding to $L \subseteq \mathbb{R}^2$ closed curve we have

$$g_{\alpha\beta} = \varepsilon_{\alpha\beta} \exp\left(\frac{c_{\alpha\beta}}{i\hbar}\right) g^{(0)}_{\alpha\beta} : W_L\big|_{U_\alpha\cap U_\beta} \longrightarrow W_L\big|_{U_\alpha \cap U_\beta}$$

$\varepsilon_{\alpha\beta} \in \check{H}^1(L, \{e^{\frac{k i\pi}{4}}\}_{k\in\mathbb{Z}})$ the maslov class of L

$W_L\big|_{U_\alpha\cap U_\beta} = |\Omega|^{1/2}(L) \otimes$ scalars

On $U_\alpha \cap U_\beta$: $\left.\begin{array}{l} \xi = \varphi_\alpha'(x) \\ x = \varphi_\beta'(x) \end{array}\right\}$ depend on signs $\operatorname{sign}(\varphi_\alpha''(x))$

$$c_{\alpha\beta} = [\xi dx|_L] \in \check{H}^1(L; \mathbb{R})$$

$g^{(0)}_{\alpha\beta} = 1 + \hbar \dots$ $g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma}$

↑ cohomologous to 1 $(1 + \hbar \rho_{\alpha\beta} + \hbar^2 \dots)(1 + \hbar \rho_{\beta\gamma} + \hbar^2 \dots) = 1 + \hbar \rho_{\alpha\gamma} + \hbar^2 \dots$

$\Longrightarrow \rho_{\alpha\beta} + \rho_{\beta\gamma} = \rho_{\alpha\gamma}$ but $\rho_{\alpha\beta} \in C^\infty(L)$ w/c is a fine sheaf

so $\rho_{\alpha\beta} = \rho_\alpha - \rho_\beta$.

$\to g^{new}_{\alpha\beta} = (1 + \hbar \rho_\alpha \cancel{+\dots})\, g^{(0)}_{\alpha\beta} (1 + \hbar \rho_\beta) = 1 + \hbar^2$

iterating $1 + \hbar^3$, this kills $g^{(0)}_{\alpha\beta}$.

Generalization to R^n

$$L = \begin{cases} \xi_1 = \Phi_1(X) \\ \vdots \\ \xi_n = \Phi_n(X) \end{cases}$$

by analogy
ξ_i act by $i\hbar \frac{\partial}{\partial x} + \Phi_i(x)$
x_i act by x_i

$$\Longrightarrow [\xi_j, x_k] = i\hbar \delta_{jk}$$

$$[x_i, x_j] = 0$$

$$[\xi_j, \xi_k] = i\hbar(\partial_j \Phi_k - \partial_k \Phi_j)$$

But $[\xi_j, \xi_k] = 0 \Longrightarrow \Phi_k = \frac{\partial}{\partial x_k}\varphi$.

Equivalently,

$$\sum d\xi_k \wedge dx_k |_L = \sum d\Phi_k \wedge dx_k = d\sum \Phi_j dx_j = d^2\varphi = 0$$

Definition: L is Lagrangian if $\dim L = 2$, $\omega|_L = 0$

Example:

$\xi_j = \frac{\partial \varphi}{\partial x_j}$ $\quad \mathbb{V}_\varphi = \{ e^{\frac{\varphi(x)}{i\hbar}} \underset{\uparrow \text{smooth}}{a(x)} \}$ supported on L_φ

locally, $L_\varphi \subseteq R^{2n}$ Lagrangian.

Example:

$x_j = \frac{\partial \psi}{\partial \xi_j}$ or other intermediate cases $\quad \begin{array}{l} x_1 = \varphi(x_2, \xi_1) \\ \xi_2 = \psi(x_2, \xi_1) \end{array}$

or $\begin{array}{l} x_2 = \varphi(x_1, \xi_2) \\ \xi_1 = \psi(x_1, \xi_2) \end{array}$

similarly, L_φ locally glue together
we get $\mathbb{V}_L - K \otimes_{\mathbb{C}[[\hbar]]} W_L$;

$W_L = |\Omega|^{1/2}(L)[[\hbar]]$ glued by $g_{\alpha\beta} = \varepsilon_{\alpha\beta} \exp\left(\frac{c_{\alpha\beta}}{i\hbar}\right)$

$[c_{\alpha\beta}] = [\sum \xi_j \, dx_j|_L]$ $\varepsilon_{\alpha\beta}$ maslov class of L.

The Quantization Package

$F \in A(\mathbb{R}^{2n}) \longmapsto Op_\hbar(F)$

$Op_\hbar(F) \circ Op_\hbar(G) \sim Op_\hbar(F * G)$ up to $O(\hbar^\infty)$

$a \in \Gamma(L, |\Omega|^{1/2} \otimes \mathcal{E}) \longmapsto U_\hbar(a)$

local system

$\varepsilon_{\alpha\beta} \cdot e^{c_{\alpha\beta}/i\hbar}$ $\quad$ $\mathbb{K}$-mod on L

We can extend this by defining

$Op_\hbar(F) \circ U_\hbar(a) \sim U_\hbar(F * a) + O(\hbar^\infty)$

A-mod structure

$F \in A(\mathbb{R}^{2n})$ then $Op_\hbar(F)$ is trace class and

$$\underbrace{\mathrm{Tr}\, Op_\hbar(F) = \frac{\overset{\text{constant}}{K_F}}{\hbar^n}\left(\int_{\mathbb{R}^{2n}} F\omega^n + \hbar \ldots\right)}_{\text{this is the package.}}$$

Globalization

introduce 2 new formal variables $\hat{x}, \hat{\xi}$.

$\Rightarrow$ new objects $f(x, \xi, \hat{x}, \hat{\xi}, \hbar)$.

$\in C^\infty(\mathbb{R}^2)[[\hat{x}, \hat{\xi}, \hbar]]$

new product $*_\omega$ in $\hat{x}, \hat{\xi}, \hbar$ (normal multiplication in coefficients)

Forms $\Omega^1 = C^\infty(\mathbb{R}^2)[[\hat{x}, \hat{\xi}, \hbar]]\langle dx, d\xi\rangle$

Let $\nabla = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}\right) dx + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \hat{\xi}}\right) d\xi = d_{dR} - \left(\frac{\partial}{\partial \hat{x}} dx + \frac{\partial}{\partial \hat{\xi}} d\xi\right)$

de Rham differential

$\nabla^2 = 0 \rightarrow$ defines a complex.

$$\Omega^0 \xrightarrow{\nabla} \Omega^1 \xrightarrow{\nabla} \Omega^2 \longrightarrow \cdots$$

Then $\ker(\nabla : \Omega^0 \longrightarrow \Omega^1) \simeq C^\infty(\mathbb{R}^2)[[\hbar]]$

(Ω^2 and the rest vanish)

$f \in \ker(\nabla : \Omega^0 \to \Omega^1) \Longrightarrow \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}\right) f = \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \hat{\xi}}\right) f = 0$

$\Longrightarrow f = f_0(x+\hat{x}, \xi + \hat{\xi}, \hbar) := \sum_{p,q} \frac{\partial_x^p \partial_\xi^q f(x,\xi)}{p!\, q!} \hat{x}^p \hat{\xi}^q$

We have $\nabla(f * g) = (\nabla f) * g + f * \nabla g$

$\longrightarrow$ $\ker \nabla$ inherits $*$ and $(C^\infty(\mathbb{R}^2)[[\hbar]], *)$

<u>Recall</u>:

$C^\infty(\mathbb{R}^2)\underbrace{[[\hat{x}, \hat{\xi}, \hbar]]}_{\text{Weyl product} *}$ and differential forms

$\Omega^1 = C^\infty(\mathbb{R}^2)[[\hat{x}, \hat{\xi}, \hbar]]\langle dx, d\xi \rangle$

$\nabla = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}\right) dx + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \hat{\xi}}\right) d\xi \quad , \quad \nabla^2 = 0$

$f, g \in \Omega^\cdot \ , \ \nabla(f * g) = (\nabla f) * g + (-1)^{|f|} f * \nabla g$

($*$ is extended to higher forms)

$H^0(\nabla) \cong C^\infty(\mathbb{R}^2)[[\hbar]]$ w/ $*$-product in x, ξ

$[x + \hat{x}, \xi + \hat{\xi}] = -i\hbar$

$H^i(\nabla) = 0 \ , \ i > 0$

<u>Proof</u>:

$\mathbb{C}[[\hat{x}, \hat{\xi}, \hbar]]$ is graded by $|\hat{x}| = |\hat{\xi}| = 1, \ |\hbar| = 2$

∇ written in terms of its homogenous parts, the leading term is

$$-\left(\frac{\partial}{\partial \hat{x}}\,dx + \frac{\partial}{\partial \hat{\xi}}\,d\xi\right)$$

the de Rham differential of $\mathbb{C}[\![\hat{x},\hat{\xi}]\!][\![\hbar]\!]$. The cohomology of this is already $C^\infty(\mathbb{R}^2)[\![\hbar]\!]$. $\square$

<u>Derivations of $\mathbb{C}[\![\hat{x},\hat{\xi},\hbar]\!]$</u>

<u>Lemma</u> Any derivation of $\mathbb{C}[\![\hat{x},\hat{\xi},\hbar]\!]$ is of the form
$f \longmapsto \frac{1}{i\hbar}[F,f]$ for some $F \in \mathbb{C}[\![\hat{x},\hat{\xi},\hbar]\!]$.
commutator

<u>Remark</u>: $Z(\mathbb{C}[\![\hat{x},\hat{\xi},\hbar]\!]) = \mathbb{C}[\![\hbar]\!] \Longrightarrow \mathfrak{g} = \operatorname{Der}\mathbb{C}[\![\hat{x},\hat{\xi},\hbar]\!] \cong \dfrac{\frac{1}{i\hbar}\mathbb{C}[\![\hat{x},\hat{\xi},\hbar]\!]}{\frac{1}{i\hbar}\mathbb{C}[\![\hbar]\!]}$.

Let $\hat{A} = (\mathbb{C}[\![\hat{x},\hat{\xi}]\!][\![\hbar]\!], *_{weyl})$, $\mathfrak{g} = \operatorname{Der}\hat{A}$, $\tilde{\mathfrak{g}} = \frac{1}{i\hbar}\hat{A}$.

$\rightarrow \quad 0 \longrightarrow \frac{1}{i\hbar}\mathbb{C}[\![\hbar]\!] \longrightarrow \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$

<u>Question</u>: Are there other connections ∇ s.t. $\nabla^2 = 0$ and
$\nabla(f*g) = (\nabla f)*g + (-1)^{|f|} f * \nabla g$.

Suppose $\nabla' = \nabla + A$, A a 1-form in x,ξ w/ values in $\mathfrak{g}$.
We further assume the ff.

Remark:

(1) $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ are also graded.

$$\mathfrak{g} = \frac{1}{i\hbar}\big(\underset{1}{A^0} \oplus \underset{\hat{x},\hat{\xi}}{A^1} \oplus \underset{\hat{x}^2,\, \hat{x}\hat{\xi},\, \hat{\xi}^2,\, \hbar}{A^2} \oplus \cdots \big)$$

$$= \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \cdots$$

$$\frac{1}{i\hbar}\hat{x} = -\frac{\partial}{\partial \hat{\xi}} \qquad \frac{1}{i\hbar}\hat{\xi} = \frac{\partial}{\partial \hat{x}}$$

$$\underbrace{\frac{1}{i\hbar}\left(\frac{\hat{x}^2}{2},\ \hat{x}\hat{\xi},\ \frac{\hat{\xi}^2}{2}\right)}_{\text{Lie algebra structure, } \cong\, sp(2n).}$$

(2) Note that we have the famous sl_2-triple $\frac{1}{2}x^2,\ x\partial + \frac{1}{2},\ \frac{1}{2}\partial^2$

artifact of Weyl symmetrization

$$x\partial + \tfrac{1}{2} = \frac{x\circ\partial + \partial\circ x}{2}$$

There is a subalgebra $\mathfrak{g}_{\geq 0} \leq \mathfrak{g}$.
The additional condition is A is $\mathfrak{g}_{\geq 0}$-valued.
This is since we want to keep the leading term as is.

Lemma: $H^0(\nabla') \cong C^\infty(\mathbb{R}^2)[[\hbar]]$; $H^i(\nabla) = 0,\ i > 0$.

has induced $\star$-product (not exactly the Weyl product).

So: Bundle (trivial) of ∞-dimensional algebras $\hat{A}, *$ on $\mathbb{R}^2$

∇ a connection on this bundle w/ $\nabla^2 = 0$ and preserves $*$.

$H^0(\nabla) = C^\infty(x, \xi)[[\hbar]]$, $H^{>0} = 0 \Longrightarrow *$ is induced.

How do we generalize this to any symplectic manifold?

Subquestion: which of these things already exist on a symplectic manifold?

Let (M, ω) be a symplectic manifold. Then the bundle $(\hat{A}_M, *)$ exist already. Let us denote by $\hat{x}, \hat{\xi}$ the basis of $T^*_{(x,\xi)}M$ corresponding to a given (Darboux) system.

More generally, $T^*_x M$ symplectic vector space.

$\hat{A}_K = \widehat{Sym}(T^*_K M)[[\hbar]]$, $*_\omega$ well-defined.

$\uparrow$ symmetric powers of $T^*_K M$ completed in $T^*_K M$-adic topology and formal power series in that.

In other words, $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow Sp(2n, \mathbb{R})$ transition functions of T^*M. Let us denote by A_M the bundle of algebras w/ fiber A glued by $g_{\alpha\beta}$.

How about the connection?

Start w/ $sp(2n)$-valued connection ∇_0 on M (symplectic connection on M) i.e $\nabla_0 \omega = 0$. This can be extended to a $Der(\hat{A})$-valued connection since $sp(2n) \hookrightarrow Der(\hat{A})$.

There are two problems.

(1) $\nabla_0^2 \neq 0$ since locally $\nabla_0 = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial \xi} d\xi + A_0 \in \Omega^1(M, \mathfrak{g}_{\geq 0})$

Let $A_{-1} = -\left(\frac{\partial}{\partial \hat{x}} dx + \frac{\partial}{\partial \hat{\xi}} d\xi\right)$ locally. Show that A_{-1} is globally well-defined. Then add $\nabla_{-1} =$ apply A_{-1}. Still a problem w/ $\nabla_1 + \nabla_0$. It preserves $*$ but it is not flat. Apply what Weinstein called the steam roller method i.e flatten it.

Do it step by step. Find $A_1, A_2, \ldots$ s.t. $A_p \in \Omega^1(M, \mathfrak{g}_p)$,

$$\nabla = A_{-1} + \nabla_0 + A_1 + A_2 + \ldots \quad \text{w/ } \nabla^2 = 0.$$

Why we can do it? The leading term A_{-1} is acyclic in deg > 0.

Now, $[\nabla, \nabla] = [A_{-1}\, \nabla_0] + (\nabla_0^2 + [A_{-1}, A]) + \ldots$

At every step, to get $\nabla^2 = 0$ mod higher degrees, solve

$$[A_{-1}, A_{p+1}] = R_p \text{ at degree } p \text{ and } [A_{-1}, R_p] = 0.$$

Seems fine, but there is no guarantee that $[A_{-1}, \nabla_0] = 0$.
So we adapt $\nabla_0 \longmapsto A_0$ s.t. $[A_{-1}, A_0] = 0$ (picking ∇_0 to be torsion-free)

Liftings

$\mathbb{R}^2$: $\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}\right) dx + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \hat{\xi}}\right) d\xi \; (+A?)$

$$\begin{array}{c} \mathfrak{g}\text{- valued connection} \\ \uparrow \\ \tilde{\mathfrak{g}} = \frac{1}{i\hbar}\hat{A},\ [,] \\ \uparrow \\ \frac{1}{i\hbar}\mathbb{C}[[\hbar]] \end{array}$$

is there a $\tilde{\mathfrak{g}}$-valued connection $\tilde{\nabla} \longmapsto \nabla$. Yes.

$$\tilde{\nabla} = \left(\frac{\partial}{\partial x} - \frac{\hat{\xi}}{i\hbar}\right) dx + \left(\frac{\partial}{\partial \xi} + \frac{\hat{x}}{i\hbar}\right) d\xi$$

(one can add central elements).

$$\tilde{\nabla}^2 = -\left[\frac{\hat{\xi}}{i\hbar}, \frac{\hat{x}}{i\hbar}\right] dx\, d\xi = \frac{1}{i\hbar} dx\, d\xi = \frac{\omega}{i\hbar}$$

Conclusion: On (M, ω), $\exists \nabla$, and $\forall \nabla, \exists \tilde{\nabla}$.

Fact: Up to isomorphism, any $*$ on (M,ω) comes from such a connection $\hat{\nabla}$.

This is because $\nabla = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}\right)dx + \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \hat{\xi}}\right)d\xi$

is coordinate-independent, under coordinate changes differing from those in T^*M.

The bundle of jets ($\forall$ manifold M) $J \longrightarrow M$.

$$J_x = \{\text{jets of } C^\infty \text{ functions at } x\} = \mathbb{C}[\![\hat{x}]\!]$$

$M = \biguplus_{i\in I} U_i$, $U_\alpha \xrightarrow{\sim} \mathbb{R}^n_\alpha \subseteq \mathbb{R}^n$

$g_{\alpha\beta} = \phi_\alpha \phi_\beta^{-1} : \mathbb{R}^n_\alpha \xleftarrow{\sim} \mathbb{R}^n_\beta$ partially defined diffeomorphism

$x_\alpha = g_{\alpha\beta}(x_\beta)$

gluing: $f(x_\alpha) \Longleftrightarrow g(x_\beta) := f(g_{\alpha\beta}(x_\alpha))$ on $U_\alpha \cap U_\beta$.

On the level of Taylor series:

$$f(x_\alpha + \hat{x}) = \sum_n \frac{1}{n!} \partial_x^n f(x_\alpha)\hat{x}^n \Longleftrightarrow f(g_{\alpha\beta}(x_\beta + \hat{x}))$$

$$G_{\alpha\beta} : \mathbb{C}[\![\hat{x}]\!]_\alpha \xleftarrow{\sim} \mathbb{C}[\![\hat{x}]\!]_\beta$$

$$x_\alpha + \hat{x} \longleftarrow g_{\alpha\beta}(x_\beta + \hat{x})$$

$$\hat{x} \longmapsto g_{\alpha\beta}(x_\beta + \hat{x}) - x_\alpha$$

Explicitly,

$$\mathbb{C}[\![\hat{x}]\!]_\alpha \xrightarrow{\sim} \mathbb{C}[\![\hat{x}]\!]_\beta$$

$$\hat{x} \longmapsto g_{\alpha\beta}(x_\beta + \hat{x}) - x_\alpha$$

$$= \underbrace{\cancel{g_{\alpha\beta}(x_\beta)}}_{x_\alpha} + g'_{\alpha\beta}(x_\beta)\hat{x} + g''_{\alpha\beta}(x_\beta)\hat{x}^2 + \dots - x_\alpha$$

The connection $\nabla = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}\right) dx$ is well-defined,

$$\Omega^{\bullet}(X, J) \xrightarrow{\nabla} \Omega^{\bullet+1}(X, J), \quad \nabla^2 = 0, \quad \nabla(fg) = (\nabla f)g + f\nabla g.$$

main reason (for what) : $x + \hat{x} \longmapsto 0$

$\mathrm{Ker}(\Omega^0 \xrightarrow{\nabla} \Omega^1)$: start w/ $f \in C^\infty(X)$,
$x \longmapsto \underset{\text{at } x}{\mathrm{Taylor}}(f)$.

We now have 2 bundles:

$$\hat{A}_M = \prod_{p=0}^{\infty} \mathrm{Sym}^p T^*_M [\![\hbar]\!]$$

$$J_M[\![\hbar]\!] = \varinjlim J^p, \quad J^p = \{ = 0 \text{ up to } p^{\text{th}} \text{ power} \}$$

$$\mathrm{Sym}^p T^*_M [\![\hbar]\!] \xleftarrow{\sim} J^p / J^{p+1}$$
as bundles

<u>Remark</u>: $\forall M$, $\widehat{\mathrm{Sym}}\, T^*M \cong J_M$ as bundles.

Now, given $*$ on (M, ω), it induces $*$ on jets on J_x, $x \in M$.
∇_{can} preserves this $*$.

$$\underbrace{\underbrace{J_M[\![\hbar]\!], *}, \nabla_{\mathrm{can}} \qquad \underbrace{\overset{\widehat{\mathrm{Sym}}^{\bullet} T^*_M[\![\hbar]\!]}{\overset{\shortparallel}{\hat{A}_M}}, *_\omega}, \nabla}_{\text{isomorphic as bundles of algebras}}$$

<u>Remark</u>:

① ∇ comes from ∇_{can} under this isomorphism.

② Any cocycle w/ values on $C^\infty(M)$-module ~ 0

③ $\forall *$ on M, $\exists \nabla \rightsquigarrow \underset{\underset{\theta}{\shortparallel}}{\tilde{\nabla}^2} = \frac{1}{i\hbar}\omega + \theta_0 + \dots$

$d\theta = 0$

Theorem:

$$\{* \text{ up to isomorphism}\} \Longleftrightarrow [\theta] \in \frac{1}{i\hbar}\omega + H^2(M,\mathbb{C})[[\hbar]]$$

Theorem: The $*$ w/ $\theta = \frac{1}{i\hbar}\omega$ comes from differential operators on $|\Omega_X|^{1/2}$.

Modules and flat connections

$A = C^\infty(x,\xi)[[\hbar]]$ $\quad V_0 = C^\infty(x)[[\hbar]]$

$\uparrow$ supported on $L = \{\xi = 0\}$.

actions:

x via multiplication by x

ξ via $i\hbar \frac{\partial}{\partial x}$

$\hat{A}$ bundle of algebras on $M = \mathbb{R}^2$

$\hat{\mathbb{V}}$ bundle of $\hat{A}|_L$-modules on L.

$\nabla_{\mathbb{V}}^2 = 0$

Local sections of $\hat{\mathbb{V}}$: $f(x, \hat{x}, \hbar)$

$\nabla_{\mathbb{V}} = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial \hat{x}}\right)dx$ $\quad$ $\hat{x}$ via multiplication by $\hat{x}$
$\hat{\xi}$ via $i\hbar \frac{\partial}{\partial \hat{x}}$

$$\nabla_{\mathbb{V}}(a \cdot v) = (\nabla_A|_L)(a) \cdot v + a \cdot \nabla_{\mathbb{V}} v$$

Nest-Tsygan:

$L \subset T^*X \longmapsto$ Bundle of $\hat{A}_{T^*X}|_L$-modules $\hat{\mathbb{V}}_L$, $\nabla_{\mathbb{V}}^2 = 0$

In particular, $\Omega^0(L, \hat{\mathbb{V}}_L)^{\nabla\text{-inv}} = \mathbb{V}_L$ is an A_M-module.

Re-glue $\mathbb{K} \otimes_{\mathbb{C}[[\hbar]]} \mathbb{V}_L$ along $g_{\alpha\beta} = \varepsilon_{\alpha\beta} \exp \frac{c_{\alpha\beta}}{i\hbar}$. we get asymptotics of Lagrangians distribution on X w/ $WF = L$.

A_M is a sheaf of algebras on (M, ω)
coming from $(\hat{A}_M, \nabla_M)$ bundle of algebras $\nabla^2 = 0$.
Fibers : $\mathbb{C}[[\hat{x}, \hat{\xi}, \hbar]]$, $*_\omega$

$$A_M \hookrightarrow \Omega^0(M, \hat{A}_M) \xrightarrow{\nabla} \Omega^1(M, \hat{A}_M) \longrightarrow$$
exact.

(sheaf of) A_M- modules $\quad \mathbb{V} = \ker(\nabla : \Omega^0(M, \mathbb{V}) \longrightarrow \Omega^1(M, \mathbb{V}))$

$\uparrow$

Bundles of $\hat{A}_M$-modules $(\mathcal{V}, \nabla_{\mathcal{V}})$ w/ compatibility

$$\nabla_{\mathcal{V}}(a \cdot v) = \nabla_A a \cdot v \pm a \cdot \nabla_{\mathcal{V}} v$$

Actually, we have not quite that :

$L = \{x = 0\} \quad \mathbb{V}_L = C^\infty(x)[[\hbar]]$ sheaf on L

$\hat{\mathbb{V}}_L = C^\infty(x)[[\hat{x}, \hbar]]$ on L

$\mathbb{V}_L := \ker(\Omega^0(L, \hat{\mathbb{V}}_L) \longrightarrow \Omega^1(L, \hat{\mathbb{V}}_L))$ mod $\ker(\Omega^0(L, \hat{A}_M) \to \Omega^1(L, \hat{A}_M))$

bundle of $\hat{A}_M$-modules

Quantization Package

$L \subseteq T^*X$

$\mathbb{V}_L = \Gamma(L, |\Omega(L)|^{1/2}) \otimes \mathbb{K}[[\hbar]]$ twisted by 1-cocycle $\epsilon_{\alpha\beta}\, e^{c_{\alpha\beta}/i\hbar}$.

$$\downarrow$$

$$\left\{ \sum_{k=0}^{\infty} e^{\alpha_k / i\hbar} \right\}_{\alpha_k \in \mathbb{R}} \quad \text{Novikov ring}$$

$\mathbb{V}_L^{\circ}$ the same module w/o $\hbar$.

$\mathbb{V}_L \ni a \longmapsto u_\hbar(a)$ family of distributions on X.

Consider X, Y, Z, $Y^{\alpha} = (Y, -\omega)$.

$L_1 \subset X \times Y^{\alpha}$, $L_2 \subset Y \times Z^{\alpha}$

$$L_1 \circ L_2 \subset X \times Z^{\alpha}$$
$$\| $$
$$\left\{ (x,z) \mid \exists y \text{ s.t. } \begin{array}{l} (x,y) \in L_1, \\ (y,z) \in L_2 \end{array} \right\}$$

$L_1 \times L_2 \subset X \times Y^{\alpha} \times Y \times Z$

intersect w/ $\Delta \subseteq Y^{\alpha} \times Y$ then project $X \times Z^{\alpha}$.

denote this by $L_1 \Delta L_2$:

$u_\hbar(a)(x,y)$ $\qquad\qquad$ $u_\hbar(b)(\cancel{x,y})$ y,z $\qquad\qquad$ $a \in |\Omega_{L_1}|^{1/2}$

$b \in |\Omega_{L_2}|^{1/2}$

$$v(x,z) := \int u_\hbar(a)(x,y)\, u_\hbar(b)(\overset{y,z}{\cancel{x,y}})\, dy$$

We hope $v(x,z) = u_\hbar(a * b)$.

Thus, we want to have in our quantization package

$$\star : \mathbb{V}_{L_1} \otimes \mathbb{V}_{L_2} \longrightarrow \mathbb{V}_{L_1 \circ L_2}$$

s.t.

$$\int u_\hbar(a)\, u_\hbar(b)\, dy \sim u_\hbar(a \star b) \mod \hbar^\infty .$$

$$\|?$$

$$\frac{c}{\hbar^m} \int \sum \hbar^p D_p(a,b)\, dy$$

bidiff. op.

projection along $L_1 \Delta L_2 \longrightarrow L_1 \circ L_2$.

Example:

$$\underbrace{T^*X \times T^*X}_{K(x,y)} \times \underset{f(y)}{pt}$$

$$u_\hbar(K(x,y)) \star u_\hbar(f) = A \cdot u_\hbar(f)$$

operator w/ kernel K

$$\underbrace{pt \times (T^*(X \times X))}_{K(x,y)} \times pt$$

$$\underbrace{(T^*(X \times X)) \times pt}_{\delta(x-y) = \text{kernel of identity}}$$

$$K(x,y) \star \delta(x-y) = \operatorname{Tr} A$$

One approach to index:

Sometimes we have $L \subset M \times M^\alpha$ s.t. $L \circ L = L$, $\Delta \subset M \times M^\alpha$.

$Z \subset M$ coisotropic of codimension $d \leq n$, $\mathcal{F}$ characteristic foliation, $L = \{\{x, y\}$ same leaf$\}$.

$$Z = \{ \xi_1 = \dots = \xi_k \}$$

leaves: $\begin{cases} x_1 = c_1 \\ \vdots \\ x_k = c_k \end{cases}$

quotient: $x_{k+1}, \dots, x_n, \xi_{x+1}, \dots, \xi_n$.

$a(x,y) \in \mathbb{V}_L$, $L \subset T^*X$

$K_\hbar(x,y) = u_\hbar(a)$ distribution on $X \times X$

$D_\hbar = \int K_\hbar(x,y) ?(y) dy$

when Fredholm,

$$\text{index}(D_\hbar) = + \operatorname{Tr}\left(e^{-D_\hbar^* D_\hbar} - e^{-D_\hbar D_\hbar^*}\right)$$

$$= \lim_{\hbar \to 0} \operatorname{Tr}\left(e^{-D_\hbar^* D_\hbar} - e^{-D_\hbar D_\hbar^*}\right)$$

$$= \tau(P - Q)$$

Hochschild Homology

$T : A/[A,A] \longrightarrow k$

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}$$

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{i-1}$$

$$\begin{tikzcd}
{} \arrow[r, "b"'] & A \otimes \bar{A}^{\otimes 2} \arrow[r, "b"'] \arrow[l, bend right, "B"'] & A \otimes \bar{A} \arrow[r, "b"'] \arrow[l, bend right, "B"'] & A \arrow[l, bend right, "B"']
\end{tikzcd} , \quad \bar{A} = A/k$$

$CC_\bullet^-(A) = C_\bullet(A)[[u]]$

$b + uB$

$$\begin{tikzcd}
C_0 \arrow[r] & C_1 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & {} \\
& C_0 \arrow[r] & C_1 \arrow[r] \arrow[d] & {} \\
& & C_0 \arrow[r] & {}
\end{tikzcd}$$

$CC_\bullet^{per}(A) = C_\bullet(A)((u))$

$$\begin{tikzcd}
CC_\bullet^- \arrow[r, hook] \arrow[d] \arrow[dd, bend right] & CC_\bullet^{per} \arrow[ddl, bend left] \\
C_\bullet \arrow[d] & \\
k &
\end{tikzcd}$$

Theorem: $CC_\bullet^{per}(A) \xrightarrow[\sim]{quis} \Omega^\bullet(X)((u))$

$A = C^\infty(X)$ $\qquad\qquad ud_R$

$$P^2 = P \longmapsto ch(P) = P + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!} \left(P - \tfrac{1}{2}\right) \otimes \overbrace{P \otimes \cdots \otimes P}^{2k} \cdot u^k \in CC_0^{per}(A)$$

$$\tau(P) = \langle \tau, ch(P) \rangle.$$

cocycle $\qquad$ cycle

$$\begin{tikzcd}
{} \arrow[r] & CC_0^{per}(A) \arrow[r] \arrow[d] & CC_1^{per}(A) \arrow[r] \arrow[d, "\tau"] & CC_0^{per}(A) \arrow[r] \arrow[d] & \cdots \\
{} \arrow[r] & 0 \arrow[r] & k \arrow[r] & 0 \arrow[r] & {}
\end{tikzcd}$$

morphism of complexes

$$\prod A \otimes \bar{A}^{\otimes 2k} u^k \longrightarrow A \xrightarrow{\tau} k$$

any homotopic τ' gives the same $\langle \tau', ch(P) \rangle$.

$$HC_\bullet^{per}(A_M) \longrightarrow H^\bullet_{dR}(M)((u))$$

$$A_M \longrightarrow A_M/\hbar = C^\infty(M)$$

Theorem: (ring of scalars is $\mathbb{C}$, not $\mathbb{C}[[\hbar]]$).
Given $\mathbb{C}$-algebra A_0, $A = A_0[[\hbar]]$

$$a * b = ab + \hbar \ldots$$

$$HC_\bullet^{per}(A) \xrightarrow[proj]{\sim} HC_\bullet^{per}(A_0)$$

Goodwillie '85.

$$\Longrightarrow \text{index}(D) = \int_M \alpha \wedge ch(P_D|_{\hbar=0}) = \int_M \alpha \wedge ch(P_{\sigma(D)})$$

principal symbol of D.

It remains to show that $\alpha = Td(X)$.

<u>Rephrase</u>:

$$\text{index}(D) = \int_{T^*X} (\text{something w/ compact support})(D).$$

<u>Why modules?</u>

Let $\mathcal{D}_X :=$ sheaf of algebras of holomorphic differential operators on X.

$M^\cdot$ = sheaf of $\mathcal{D}_X$-modules

$$R\mathcal{H}om_{\mathcal{D}_X}(M^\cdot, N^\cdot) = \mathcal{E}xt^\cdot_{\mathcal{D}_X}(M^\cdot, N^\cdot)$$

$\mathcal{D}_\hbar$-modules : $\mathcal{O}_X[i\hbar\partial x_1, \dots, i\hbar\partial x_n][\hbar] \longleftarrow$ Rees ring of $\mathcal{D}_X$

$\mathcal{D}_X^\hbar$-modules

$$\downarrow$$

$$A_{T^*X}$$

$V_0 = C^\infty(x)$ or $\mathcal{O}_{hol}(x)$ $\quad x, i\hbar\partial_x \quad Ext^\cdot_{\mathcal{D}_X}(\mathcal{O}_X, V)$

$$R^{-1} \longrightarrow R^0$$

$$0 \longrightarrow \mathcal{D}_X \xrightarrow{\cdot\partial_x} \mathcal{D}_X \longrightarrow \mathcal{O}_X \quad \text{free resolution}$$

$$\mathcal{H}om_{\mathcal{D}_X}(R^\cdot, V)$$

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, V) = V$$

$$0 \longleftarrow V_1 \xleftarrow{\partial_x} V_0 \longleftarrow 0$$

$$V_1\,\partial x \xleftarrow{\partial x} V_0 \qquad DR^{\cdot}(V)$$

<u>Example</u>: $M^{\cdot}$ (or $V^{\cdot}$)

given $D: \mathcal{O}_X \longrightarrow \mathcal{O}_X$
diff. operators

$$0 \longrightarrow \mathcal{O}_X^N \xrightarrow{D} \mathcal{O}_X^N \longrightarrow 0$$

if applicable, $\operatorname{ind}(D) = \chi(\mathcal{O} \xrightarrow{D} \mathcal{O})$

$$M^{\cdot} \qquad \mathcal{D}_X^N \xrightarrow{\cdot D} \mathcal{D}_X^N$$

complex of free left $\mathcal{D}_X$-modules

<u>Remark</u>

(1) $Ext^{\cdot}_{\mathcal{D}_X}(\ ,\ )$ captures DR

(2) $Ext^{\cdot}_{A_h}(\ ,\ )$ captures geometry of M, L to some extent